

The Intriguing Structure of Non-geometric Frames in String Theory

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Abstract

Non-geometric frames in string theory are related to the geometric ones by certain local $O(D, D)$ transformations, the so-called β -transforms. For each such transformation, we show that there exists both a natural field redefinition of the metric and the Kalb-Ramond two-form as well as an associated Lie algebroid. We furthermore prove that the all-order low-energy effective action of the superstring, written in terms of the redefined fields, can be expressed through differential-geometric objects of the corresponding Lie algebroid. Thus, the latter provides a natural framework for effective superstring actions in non-geometric frames. Relations of this new formalism to double field theory and to the description of non-geometric backgrounds such as T-folds are discussed as well.

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1 Introduction

One of the celebrated features of string theory is that after quantizing the closed string, one generically finds a massless mode in the spectrum, which has all the properties of a graviton. Another important aspect is that the graviton is accompanied by two additional massless excitations, namely the Kalb-Ramond field and the dilaton. The leading-order dynamics of these fields is governed by an effective action containing the Einstein-Hilbert term for gravity and the kinetic terms of the Kalb-Ramond field and the dilaton. This action in the so-called geometric frame has two types of local symmetries, namely it is invariant under

diffeomorphisms of the space-time coordinates and under gauge transformations of the Kalb-Ramond field. String theory furthermore provides higher-order α' -corrections which involve e.g. higher powers of the Riemann tensor.

String theory transcends the usual notions of field theory by the existence of new transformations where string momentum and winding modes are exchanged. These so-called T-dualities are crucial and have been a valuable guide for the detection of new structures in string theory, such as mirror symmetry or D-branes. Moreover, this T-duality, via the Buscher rules, acts non-trivially on the metric, the Kalb-Ramond form and the dilaton. In particular the metric and the Kalb-Ramond field become closely intertwined. For a compactification on a D -dimensional torus, the D^2 -dimensional moduli space becomes $O(D, D; \mathbb{R})/O(D) \times O(D)$ which in string theory is further divided by the T-duality group $O(D, D; \mathbb{Z})$.

In view of this, it is a natural question whether one can implement these $O(D, D)$ transformations, whose origin lies in the decoupling of left- and right-movers on the string world-sheet, directly in the space-time effective action of string theory. Indeed, following some earlier work [1, 2], two frameworks were developed where the $O(D, D)$ transformations¹ play a crucial role, namely generalized geometry [3, 4, 5, 6] and double field theory (DFT) [7, 8, 9, 10, 11]. In the first approach, the concept of Riemannian geometry is extended from the tangent bundle TM to the generalized tangent bundle $TM \oplus T^*M$, whereas in the second the dimension of the space is doubled by including winding coordinates subject to certain constraints. For the latter construction, this admits a manifest global $O(D, D)$ invariance of the action, so in particular, the action is manifestly invariant under T-duality transformations. The fundamental object in both approaches is a generalized metric which combines the usual metric and Kalb-Ramond field. The two local symmetries, diffeomorphisms and B -field gauge transformations, sit inside a subgroup of $O(D, D)$. Their complement in $O(D, D)$ contains so-called (local) β -transforms, which lead out of the usual geometric frame of string theory. Therefore, applying a local β -transform to the geometric frame leads to what we call a *non-geometric frame*.

The existence of *non-geometric backgrounds* can be seen by analyzing the action of T-duality on the simple background of a flat three-dimensional torus with a constant H -flux [12]. Applying successive T-dualities, this H -flux is first mapped to a geometric flux [13] and by a second T-duality to the non-geometric Q -flux [14, 15, 16]. The latter background can be understood as a T-fold [17], where the transition functions between two charts involve T-duality transformations. A third T-duality is beyond the scope of the Buscher rules, and both non-commutative geometry [18, 19, 20] and conformal field theory [21, 22, 23, 24, 25] hint towards a non-associative structure. The effect of T-duality on brane solutions

¹If not otherwise specified, the short-hand notation $O(D, D)$ stands for *local* $O(D, D)$ transformations, i.e. those which non-trivially depend on the coordinates.

has been analyzed recently in [26].

Since in DFT a global $O(D, D)$ symmetry is manifest, the first-order effective action in at least a subset of these non-geometric frames is also described by it. What has been puzzling is that the DFT action cannot be straightforwardly interpreted as the Einstein-Hilbert action of some $O(D, D)$ covariant differential geometry [27, 28]. The problem is that the notions of torsion and curvature have to be changed to make them tensors so that they do not satisfy some of the usual properties of Riemannian geometry – the Levi-Civita connection is not unique and the curvature has more symmetries compared to the usual case. That is not a major problem in itself, but higher-order α' -corrections involve the full Riemann tensor, so it is not clear how to describe these. The analogous situation has also been encountered in attempts to generalize DFT to M-theory by making the U-duality groups manifest (see e.g. [29, 30, 31]).

In this paper we follow a slightly less ambitious approach which is motivated by the recent studies of effective actions in non-geometric frames. In [32, 33, 34] the geometric action was redefined using a non-geometric frame. This gave an action containing the metric and a bi-vector field β as the dynamical fields and involved a new type of Ricci scalar. In [35, 36] the starting point was the abstract structure of a Lie algebroid and, for a special case, a differential geometry was developed whose Einstein-Hilbert term could be related to the Einstein-Hilbert term in the geometric frame via a field redefinition. At that stage these two approaches might look a bit ad hoc.

We clarify the conceptual status of these two actions and show that they fit into a larger picture in which mathematically the differential geometry of Lie algebroids plays an important role. The starting point is the geometric frame. Then, applying a general local $O(D, D)$ transformation, from its action on the generalized metric we can read off a field redefinition for the metric and B -field. For the geometric subgroup of diffeomorphisms and gauge transformations this reduces to the familiar form, however β -transformations give a non-trivial redefinition. With the field redefinition at hand, one can express the action in terms of these new field variables. We show that for each non-geometric local $O(D, D)$ transformation this action is based on nothing else than the differential geometry of a corresponding Lie algebroid, whose defining data can also be directly read off from the $O(D, D)$ matrix.

Thus, this allows us to describe the low-energy effective action of string theory in every non-geometric frame in terms of a (generalized) differential geometry where, opposed to DFT, the definitions of torsion and curvature still keep the familiar forms. Therefore, there also exists a Riemann tensor and it is clear how higher-order α' -corrections are described in these non-geometric frames. To emphasize it again, we are not, as in DFT, covariantizing part of the entire $O(D, D)$ symmetry, but provide a uniform description of the string actions in any non-geometric frame in terms of a new differential geometry. In each such frame, the action only enjoys the usual diffeomorphism and gauge symmetries.

Working still in the framework of generalized geometry, in contrast to DFT, we do not have the local symmetries related to the winding-coordinate dependence of the usual and winding diffeomorphisms. As we will see, as a consequence, the description of global non-geometric backgrounds, like the constant Q -flux example, is not possible within a single frame.²

This paper is organized as follows: In section 2 we recall some basics notions of generalized geometry and show that every $O(D, D)$ transformation naturally induces a corresponding field redefinition. For β -transformations, this goes beyond the realm of differential geometry. Two examples are presented, which were previously discussed in the literature. We point out that the mathematical framework, capturing the structure of the geometry in the redefined variables, is based on so-called Lie algebroids. In section 3, after an introduction to Lie algebroids we outline the corresponding differential geometry which by construction is covariant under diffeomorphisms. Then we discuss how one can define also a Lie algebroid from an $O(D, D)$ transformation. In section 4, we generally prove that the differential geometry in the redefined variables is nothing else than the differential geometry of the corresponding Lie algebroid. The final NS-NS action in the redefined variables is presented and shown to be invariant under diffeomorphisms and the analog of B -field gauge transformations in the new variables. In section 5 we discuss further aspects of this formalism, namely we clarify the relation to double field theory, the extension to superstring effective actions to higher-order α' -corrections and provide the tree-level equations of motions in each non-geometric frame. Finally, we elaborate on the relation and distinction between what we have called non-geometric frames, which is a choice of variables, and the description of global non-geometric string backgrounds. The upshot is that, in a non-geometric frame, in each patch a non-geometric background might take a very simple form. However, the transition functions are still given by transformations, i.e. β -transforms, which are not a symmetry of the action in each patch.

2 Generalized geometry

In this section, we show that for every local $O(D, D)$ transformation a corresponding field redefinition can be deduced. In order to do so, we start by recalling some basics on generalized geometry.

²Note that in DFT, a non-geometric background can be characterized by the appearance of winding coordinates either directly in the dependence of the DFT metric (as in the toroidal constant R -flux example) or in the transition functions between two patches (as in the toroidal constant Q -flux example).

2.1 $O(D, D)$ transformations and the generalized metric

Let us briefly introduce $O(D, D)$ transformations as well as the concept of a generalized metric. For more details, we refer the reader to [5].

Basics on generalized geometry

We consider a D -dimensional manifold M together with the so-called generalized tangent bundle $E = TM \oplus T^*M$. The elements in E will be denoted by the formal sum $(X + \xi) \in \Gamma(E)$, where $X \in \Gamma(TM)$ is a vector field and $\xi \in \Gamma(T^*M)$ is a one-form. The natural bilinear form on the bundle E is

$$\langle X + \xi, Y + \zeta \rangle = \xi(Y) + \zeta(X), \quad (2.1)$$

where the action of say $\xi = \xi_\alpha e^\alpha$ on $Y = Y^a e_a$ is given by $\xi(Y) = \xi_a Y^a$. The bilinear form (2.1) can also be described in terms of a $2D \times 2D$ matrix³

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \quad (2.2)$$

The transformations \mathcal{M} which leave (2.2) invariant, that is

$$\mathcal{M}^t \eta \mathcal{M} = \eta, \quad (2.3)$$

constitute the group $O(D, D)$. A general matrix $\mathcal{M} \in O(D, D)$ can be decomposed into four $D \times D$ matrices as follows

$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (2.4)$$

and equation (2.3) then yields three independent constraints on the submatrices, namely

$$\begin{aligned} a^t c + c^t a &= 0, \\ b^t d + d^t b &= 0, \\ b^t c + d^t a &= \mathbb{1}. \end{aligned} \quad (2.5)$$

Note that in our conventions, the $O(D, D)$ matrix (2.4) acts on a tuple $(X^a, \xi_\alpha)^t$, with $X = X^a e_a$ a vector field and $\xi = \xi_\alpha e^\alpha$ a one-form. Therefore, the index structure of the submatrices in (2.4) is

$$a^a{}_b, \quad b^{a\beta}, \quad c_{\alpha b}, \quad d_\alpha{}^\beta. \quad (2.6)$$

³Explicitly, this means that (2.1) can be written as $\langle X + \xi, Y + \zeta \rangle = \begin{pmatrix} X^a \\ \xi_\alpha \end{pmatrix}^t \begin{pmatrix} 0 & \delta_a{}^\beta \\ \delta^{\alpha b} & 0 \end{pmatrix} \begin{pmatrix} Y^b \\ \zeta_\beta \end{pmatrix}$.

The generalized metric

Let us now combine the metric G_{ab} of the manifold M and the antisymmetric Kalb-Ramond field B_{ab} into the so-called generalized metric

$$\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \quad (2.7)$$

Note that \mathcal{H} satisfies $(\eta\mathcal{H})^2 = \mathbb{1}$, and that elements of the group $O(D, D)$ act on the generalized metric by conjugation

$$\hat{\mathcal{H}} = \mathcal{M}^t \mathcal{H} \mathcal{M}, \quad \mathcal{M} \in O(D, D). \quad (2.8)$$

Since in general the metric \mathcal{H} depends non-trivially on the coordinates $x \in M$ through G and B , we allow for an x -dependence in the transformation matrix, i.e. we consider local $O(D, D)$ transformations $\mathcal{M}(x)$. However, to keep our formulas readable, we mostly omit the explicit coordinate dependence in the following.

Since G is symmetric and B is antisymmetric, a priori \mathcal{H} contains D^2 free parameters. But because $O(D, D)$ has $2D^2 - D$ free parameters, it is suggestive that there exists a subgroup of $O(D, D)$ which leaves \mathcal{H} invariant. These automorphisms are represented by the matrices

$$\begin{aligned} \mathcal{M}_{\text{auto}}^{(1)} &= \begin{pmatrix} \mathcal{O}_1 & 0 \\ B\mathcal{O}_1 - (\mathcal{O}_1^t)^{-1}B & (\mathcal{O}_1^t)^{-1} \end{pmatrix}, \\ \mathcal{M}_{\text{auto}}^{(2)} &= \begin{pmatrix} -G^{-1}(\mathcal{O}_2^t)^{-1}B & G^{-1}(\mathcal{O}_2^t)^{-1} \\ G\mathcal{O}_2 - BG^{-1}(\mathcal{O}_2^t)^{-1}B & BG^{-1}(\mathcal{O}_2^t)^{-1} \end{pmatrix}, \end{aligned} \quad (2.9)$$

where $\mathcal{O}_{1,2} \in O_G(D)$.⁴ It can be checked explicitly that transformations of the form (2.9) preserve the generalized metric (2.7).

$O(D, D)$ transformations

Let us now turn to other subgroups of $O(D, D)$, which will become important in our subsequent discussion.

- The geometric subgroup $G_{\text{geom}} \subset O(D, D)$ consists of the group of diffeomorphisms $G_{\text{diffeo}} \subset G_{\text{geom}}$ characterized by

$$\mathcal{M}_{\text{diffeo}} = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}, \quad (2.10)$$

⁴We consider the local orthogonal group with respect to the metric G , that is those matrices \mathcal{O} which satisfy $\mathcal{O}^t G \mathcal{O} = G$. The metric is positive definite as we are considering a Euclidean manifold M .

diffeomorphisms	$\mathcal{M}_{\text{diffeo}} = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & (\mathbf{A}^t)^{-1} \end{pmatrix}$	$G_{\text{diffeo}} \subset G_{\text{geom}} \subset O(D, D)$
B -transforms	$\mathcal{M}_{\mathbf{B}} = \begin{pmatrix} \mathbb{1} & 0 \\ -\mathbf{B} & \mathbb{1} \end{pmatrix}$	$G_{\mathbf{B}} \subset O(D, D)$
β -transforms	$\mathcal{M}^{\beta} = \begin{pmatrix} \mathbb{1} & -\beta \\ 0 & \mathbb{1} \end{pmatrix}$	$G_{\beta} \subset O(D, D)/G_{\text{geom}}$

Table 1: Summary of $O(D, D)$ transformations discussed in the main text.

with \mathbf{A} an invertible $D \times D$ matrix. The matrices (2.10) give rise to diffeomorphism transformations of the metric and B -field, which can be seen from

$$\mathcal{H}(A^t G A, A^t B A) = \mathcal{M}_{\text{diffeo}}^t \mathcal{H}(G, B) \mathcal{M}_{\text{diffeo}}. \quad (2.11)$$

- The group of so-called B -transforms $G_{\mathbf{B}} \subset O(D, D)$ is given by matrices

$$\mathcal{M}_{\mathbf{B}} = \begin{pmatrix} \mathbb{1} & 0 \\ -\mathbf{B} & \mathbb{1} \end{pmatrix}, \quad (2.12)$$

where \mathbf{B} is an antisymmetric $D \times D$ matrix. For $\mathbf{B} = d\Lambda$, these B -transforms describe gauge transformations of the Kalb-Ramond field. Indeed, one can check that

$$\mathcal{H}(G, B + d\Lambda) = \mathcal{M}_{d\Lambda}^t \mathcal{H}(G, B) \mathcal{M}_{d\Lambda}. \quad (2.13)$$

The latter transformations therefore belong to the geometric subgroup G_{geom} , in particular, $G_{d\Lambda}$ is a normal subgroup of G_{geom} , i.e. $G_{\text{geom}} = G_{d\Lambda} \rtimes G_{\text{diffeo}}$.

- Finally, the so-called β -transforms G_{β} are contained in the complement $O(D, D)/G_{\text{geom}}$ and take the form

$$\mathcal{M}^{\beta} = \begin{pmatrix} \mathbb{1} & -\beta \\ 0 & \mathbb{1} \end{pmatrix}, \quad (2.14)$$

whose action on \mathcal{H} is not just given by diffeomorphisms or gauge transformations, but goes beyond the geometric frame. Hence, the resulting new frame is called a *non-geometric frame*.

In table 1, we have summarized the three types of transformations discussed in this paragraph.

2.2 $O(D, D)$ -induced field redefinition

As we have illustrated, the generalized metric (2.7) encodes G and B in a way that is suitable for implementing the $O(D, D)$ structure. However, a general $O(D, D)$ transformation mixes the entries of $\mathcal{H}(G, B)$ in a complicated manner. If we want to cast the transformed metric $\hat{\mathcal{H}}(G, B)$ in (2.8) into the standard form (2.7), we are required to perform a field redefinition, leading to a new metric \hat{G} and Kalb-Ramond field \hat{B} . These steps can be represented schematically as follows:

$$\begin{array}{ccccc}
 \mathcal{H}(G, B) & \xrightarrow{\mathcal{M}^t \mathcal{H} \mathcal{M}} & \hat{\mathcal{H}}(G, B) & \xrightarrow{\hat{G}(G, B), \hat{B}(G, B)} & \mathcal{H}(\hat{G}, \hat{B}) \\
 \text{generalized metric in} & & O(D, D)\text{-transformed} & & \text{generalized metric in} \\
 \text{variables } G \text{ and } B & & \text{generalized metric} & & \text{new variables}
 \end{array}$$

Therefore, at the level of the metric G and Kalb-Ramond field B , the redefinitions $\hat{G}(G, B)$ and $\hat{B}(G, B)$ are the manifestation of $O(D, D)$ transformations.

In this section, we show that for *every* $O(D, D)$ transformation of the generalized metric (2.7), one can read off a field redefinition for the metric G and two-form B . These redefinitions take a concise form and allow for a treatment in terms of so-called Lie algebroids, which will be introduced in section 3.

Field redefinition

Let us start by performing a general $O(D, D)$ transformation (2.8) on the generalized metric \mathcal{H}

$$\hat{\mathcal{H}}(G, B) = \mathcal{M}^t \mathcal{H}(G, B) \mathcal{M}. \quad (2.15)$$

With \mathcal{M} of the form shown in (2.4), we obtain the following expression for the lower-right component of $\hat{\mathcal{H}}(G, B)$:

$$\hat{\mathcal{H}}_{\text{lr}} = [d + (G - B)b]^t G^{-1} [d + (G - B)b]. \quad (2.16)$$

Comparing this with the original expression $\mathcal{H}_{\text{lr}} = G^{-1}$, we see that (2.16) should be the inverse of the new metric \hat{G} . We therefore define

$$\hat{G} = \gamma^{-1} G (\gamma^{-1})^t, \quad (2.17)$$

where the matrix γ is given by

$$\gamma = d + (G - B)b. \quad (2.18)$$

Note that, as shown in appendix A, in the case of a Euclidean metric, i.e. for G positive definite, the matrix γ is always invertible. In particular, this includes the most interesting case where only the internal space is described by a

non-geometric frame, whereas for the flat Minkowskian part one still uses the geometric frame. However, to avoid confusions, we will assume the whole space-time metric to be Euclidean in the rest of this paper.

In order to determine the redefined Kalb-Ramond field \hat{B} , it is convenient to consider the upper-right component of the generalized metric. In particular, under an $O(D, D)$ transformation \mathcal{H}_{ur} transforms as

$$\hat{\mathcal{H}}_{\text{ur}} = -\mathbb{1} + [c + (G - B)a]^t G^{-1} [d + (G - B)b]. \quad (2.19)$$

After comparing with the standard form $\mathcal{H}_{\text{ur}} = BG^{-1}$ we are led to the field redefinition

$$\hat{B} = \gamma^{-1} [\gamma \delta^t - G] (\gamma^{-1})^t, \quad (2.20)$$

with the matrix δ defined as

$$\delta = c + (G - B)a. \quad (2.21)$$

By employing the $O(D, D)$ properties (2.5), one can show that \hat{B} in (2.20) is indeed antisymmetric. The remaining components of the generalized metric can be determined from (2.17) and (2.20) via the relation $(\eta\mathcal{H})^2 = \mathbb{1}$. To summarize, an $O(D, D)$ transformation of the generalized metric \mathcal{H} gives rise to the following field redefinitions:

$$\begin{aligned} \hat{G} &= \gamma^{-1} G (\gamma^{-1})^t, & \gamma &= d + (G - B)b, \\ \hat{B} &= \gamma^{-1} [\gamma \delta^t - G] (\gamma^{-1})^t, & \delta &= c + (G - B)a. \end{aligned} \quad (2.22)$$

Remarks

Let us close our discussion of the field redefinitions with the following two remarks. First, the inverse of the relations (2.22) is given by

$$G = \hat{\gamma}^{-1} \hat{G} (\hat{\gamma}^{-1})^t, \quad B = \hat{\gamma}^{-1} [\hat{\gamma} \hat{\delta}^t - \hat{G}] (\hat{\gamma}^{-1})^t, \quad (2.23)$$

written in terms of $\hat{\delta}$ and the inverse matrix $\gamma^{-1} = \hat{\gamma}$, which can be expressed as

$$\hat{\gamma} = a^t + (\hat{G} - \hat{B})b^t, \quad \hat{\delta} = c^t + (\hat{G} - \hat{B})d^t. \quad (2.24)$$

Second, for the elements in the geometric subgroup G_{geom} , the field redefinitions (2.22) simplify considerably (see also [37]). In particular, for diffeomorphisms (2.10) we obtain

$$\hat{G} = A^t G A, \quad \hat{B} = A^t B A, \quad (2.25)$$

which is just the transformation behavior of tensors under diffeomorphisms. For gauge transformations (2.12), given by B -transforms with $\mathbf{B} = d\Lambda$, we also obtain the expected transformation properties

$$\widehat{G} = G, \quad \widehat{B} = B + d\Lambda. \quad (2.26)$$

Since under these two types of local transformations the string effective action is invariant, the field redefinitions are not transcending it. This is different for the non-geometric β -transforms, which induce a field dependent redefinition of the metric and the Kalb-Ramond field. We come back to this point below.

2.3 Examples of non-geometric frames

Let us illustrate the method introduced above by two examples. More concretely, we revisit two particular $O(D, D)$ transformations of the generalized metric (2.7) which have been discussed in the literature.

Frame I

For the first example, we consider a setting which has recently been employed in [32, 33, 34]. The matrix parametrizing the transformation of the generalized metric takes the form

$$\mathcal{M}_I = \begin{pmatrix} 0 & (G - BG^{-1}B)^{-1} \\ G - BG^{-1}B & 0 \end{pmatrix}, \quad (2.27)$$

which is indeed an $O(D, D)$ transformation since the conditions (2.3) are satisfied. The transformed metric $\widehat{\mathcal{H}}_I(G, B)$, written in terms of the original fields G and B , is then obtained as

$$\begin{aligned} \widehat{\mathcal{H}}_I &= \mathcal{M}_I^t \mathcal{H} \mathcal{M}_I \\ &= \begin{pmatrix} (G - BG^{-1}B)G^{-1}(G - BG^{-1}B) & -BG^{-1} \\ G^{-1}B & (G - BG^{-1}B)^{-1} \end{pmatrix}. \end{aligned} \quad (2.28)$$

In order to express this metric again in the form (2.7), we employ the general formulas (2.22) to arrive at the field redefinitions

$$\begin{aligned} \widehat{G} &= (\mathbb{1} + BG^{-1})G(\mathbb{1} - G^{-1}B), \\ \widehat{B} &= -(\mathbb{1} + BG^{-1})B(\mathbb{1} - G^{-1}B). \end{aligned} \quad (2.29)$$

Furthermore, it turns out to be convenient to define an antisymmetric bi-vector $\widehat{\beta}$ as follows:

$$\widehat{\beta} = \widehat{G}^{-1}\widehat{B}\widehat{G}^{-1}. \quad (2.30)$$

With the help of (2.30), we then obtain the relation

$$(G + B)^{-1} = \widehat{G}^{-1} + \widehat{\beta}, \quad (2.31)$$

so that (2.29) can alternatively be written as

$$\begin{aligned} G &= (\widehat{G}^{-1} - \widehat{\beta})^{-1} \widehat{G}^{-1} (\widehat{G}^{-1} + \widehat{\beta})^{-1}, \\ B &= -(\widehat{G}^{-1} - \widehat{\beta})^{-1} \widehat{\beta} (\widehat{G}^{-1} + \widehat{\beta})^{-1}, \end{aligned} \quad (2.32)$$

which is precisely the field redefinition employed in [32, 33, 34]. Moreover, from (2.29) we realize that the $O(D, D)$ transformation (2.27) can also be expressed as

$$\mathcal{M}_I = \begin{pmatrix} 0 & \widehat{G}^{-1} \\ \widehat{G} & 0 \end{pmatrix}. \quad (2.33)$$

Only for a background which is flat in the redefined variables, for instance a toroidal one, the transformed metric is of the form $\widehat{G}_{ab} = \delta_{ab}$.

Frame II

The second example we want to discuss has recently appeared in [35, 36]. It is characterized by an $O(D, D)$ transformation given by the following matrix

$$\mathcal{M}_{II} = \mathcal{M}_{-2B} \mathcal{M}^{\widehat{\beta}} = \begin{pmatrix} \mathbf{1} & -\widehat{\beta} \\ 2B & -\mathbf{1} \end{pmatrix}, \quad (2.34)$$

which consists of a combination of a B - and a β -transform. Note that in order for (2.34) to satisfy the $O(D, D)$ properties (2.5), we have to require $\widehat{\beta} = B^{-1}$. The generalized metric resulting from (2.34) is

$$\widehat{\mathcal{H}}_{II} = \mathcal{M}_{II}^t \mathcal{H} \mathcal{M}_{II} = \begin{pmatrix} G - BG^{-1}B & -GB^{-1} \\ B^{-1}G & -B^{-1}GB^{-1} \end{pmatrix}. \quad (2.35)$$

To make a connection to (2.7) in the standard form, we introduce a metric \widehat{g} on the co-tangent bundle T^*M as well as an antisymmetric bi-vector $\widehat{\beta}$ by

$$\widehat{g} = -B^{-1}G B^{-1}, \quad \widehat{\beta} = B^{-1}. \quad (2.36)$$

This field redefinition can formally be regarded as the Seiberg-Witten limit of (2.29), and was studied in detail in [35, 36]. In these variables, the transformed metric (2.35) is expressed as

$$\widehat{\mathcal{H}} = \begin{pmatrix} \widehat{g}^{-1} - \widehat{\beta}^{-1}\widehat{g}\widehat{\beta}^{-1} & \widehat{\beta}^{-1}\widehat{g} \\ -\widehat{g}\widehat{\beta}^{-1} & \widehat{g} \end{pmatrix}. \quad (2.37)$$

2.4 The quest for non-geometric actions

In the last two subsections, we have demonstrated how any local $O(D, D)$ transformation gives rise to a field redefinition. In the following sections, we will elaborate on the underlying structure of the low energy effective action of string theory expressed in terms of the redefined variables.

Recall that the leading order action for the metric, the Kalb-Ramond field and the dilaton in an arbitrary number of dimensions is⁵

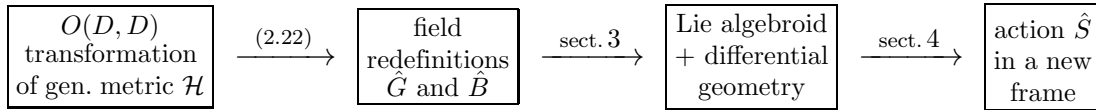
$$S = -\frac{1}{2\kappa^2} \int d^n x \sqrt{|G|} e^{-2\phi} \left(R - \frac{1}{12} H_{abc} H^{abc} + 4\partial_a \phi \partial^a \phi \right). \quad (2.38)$$

This action is manifestly invariant under diffeomorphisms and under gauge transformations $B \rightarrow B + d\Lambda$ of the Kalb-Ramond field, i.e. transformations which are encoded in the geometric group G_{geom} . However, upon performing a β -transformation, the implied field redefinition is not a symmetry of the action (2.38). Hence, in the variables corresponding to a β -transform, the action will take a different form.

Let us illustrate this observation with the non-geometric Frame II. We recall from [35, 36] that under the field redefinition (2.36) the action (2.38) changes to

$$\hat{S} = -\frac{1}{2\kappa^2} \int d^n x \sqrt{|\hat{g}|} |\hat{\beta}^{-1}| e^{-2\phi} \left(\hat{R} - \frac{1}{12} \hat{\Theta}^{abc} \hat{\Theta}_{abc} + 4\hat{g}_{ab} D^a \phi D^b \phi \right). \quad (2.39)$$

Here, a new derivative operator $D^a = \hat{\beta}^{am} \partial_m$ has been introduced, \hat{R} denotes a curvature scalar to be specified in the next section, and we have defined $\hat{\Theta}^{abc} = 3D^{[a} \hat{\beta}^{bc]}$ ⁶. In [35, 36] it has been shown that the action (2.39) can be interpreted as coming from the differential geometry of a *Lie algebroid*. In the subsequent sections of this paper, we show that this is just a particular example of a more general story. Namely, for each non-geometric frame there exists a corresponding field redefinition together with a Lie algebroid, such that the transformed action \hat{S} is governed by the corresponding differential geometry.



3 Lie algebroids

In this section, we provide some details on the mathematical structure of a Lie algebroid. Roughly speaking, a Lie algebroid is a generalization of a Lie algebra where the structure constants can be space-time dependent. In particular, the Lie

⁵For matrices, $|\dots|$ denotes the absolute value of the determinant.

⁶The anti-symmetrization of n indices includes a factor $1/n!$.

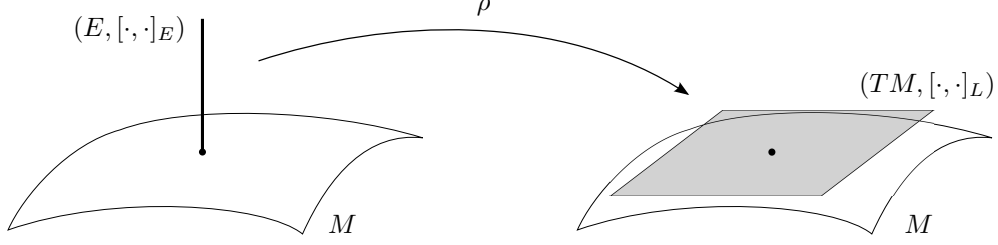


Figure 1: Illustration of a Lie algebroid. On the left, one can see a manifold M together with a bundle E and a bracket $[\cdot, \cdot]_E$. This structure is mapped via the anchor ρ to the tangent bundle TM with Lie bracket $[\cdot, \cdot]_L$, which is shown on the right.

bracket for vector fields is generalized to a bracket for sections in a general vector bundle satisfying similar properties. Lie algebroids admit a natural generalization of the usual differential geometry framework, and hence covariant derivatives, torsion and curvature tensors can be constructed. The relevance of Lie algebroids in the context of non-geometric fluxes has already been indicated in earlier work, for example in [38, 39, 29, 40].

3.1 Definition and examples

Let us introduce the concept of a Lie algebroid and illustrate this structure by two examples. To specify a Lie algebroid one needs three pieces of information:

- a vector bundle E over a manifold M ,
- a bracket $[\cdot, \cdot]_E : E \times E \rightarrow E$, and
- a homomorphism $\rho : E \rightarrow TM$ called the anchor.

A pictorial illustration for a Lie algebroid can be found in figure 1. Similar to the usual Lie bracket, we require the bracket $[\cdot, \cdot]_E$ to satisfy a Leibniz rule. Denoting functions by $f \in \mathcal{C}^\infty(M)$ and sections of E by s_i , this reads

$$[s_1, f s_2]_E = f [s_1, s_2]_E + \rho(s_1)(f) s_2, \quad (3.1)$$

where $\rho(s_1)$ is a vector field which acts on f as a derivation. If in addition the bracket $[\cdot, \cdot]_E$ satisfies a Jacobi identity

$$[s_1, [s_2, s_3]_E]_E = [[s_1, s_2]_E, s_3]_E + [s_2, [s_1, s_3]_E]_E, \quad (3.2)$$

then $(E, [\cdot, \cdot]_E, \rho)$ is called a *Lie algebroid*.⁷ Therefore, in a Lie algebroid vector

⁷If the Jacobi identity is not satisfied, the resulting structure is called a quasi-Lie algebroid.

fields and their Lie bracket $[\cdot, \cdot]_L$ are replaced by sections of E and the corresponding bracket $[\cdot, \cdot]_E$. The relation between the different brackets is established by the anchor ρ . Indeed, the requirement that ρ is a homomorphism implies that

$$\rho([s_1, s_2]_E) = [\rho(s_1), \rho(s_2)]_L. \quad (3.3)$$

Let us illustrate this construction by two examples. The first is the trivial example, while the second one will be relevant in later sections of this paper.

- Consider the tangent bundle $E = TM$ with the usual Lie bracket $[\cdot, \cdot]_E = [\cdot, \cdot]_L$. The anchor is chosen to be the identity map, i.e. $\rho = \text{id}$. Then, the conditions (3.1) and (3.2) reduce to the well-known properties of the Lie bracket, and (3.3) is trivially satisfied. Therefore, $E = (TM, [\cdot, \cdot]_L, \rho = \text{id})$ is indeed a Lie algebroid.
- As a second example, we consider a Poisson manifold (M, β) with Poisson tensor $\beta = \frac{1}{2} \beta^{ab} e_a \wedge e_b$, where $\{e_a\}$ denotes a basis of vector fields. A Lie algebroid is given by $E = (T^*M, [\cdot, \cdot]_K, \rho = \beta^\sharp)$, in which the anchor β^\sharp is defined as

$$\beta^\sharp(e^a) = \beta^{am} e_m, \quad (3.4)$$

with $\{e^a\} \in \Gamma(T^*M)$ the basis of one-forms dual to the vector field basis. The bracket $[\cdot, \cdot]_K$ on T^*M is the *Koszul bracket*, which for one-forms ξ and η is defined as⁸

$$[\xi, \eta]_K = L_{\beta^\sharp(\xi)}\eta - \iota_{\beta^\sharp(\eta)}d\xi, \quad (3.5)$$

where the Lie derivative on forms is given by $L_X = \iota_X \circ d + d \circ \iota_X$ with d the de Rham differential. The conditions (3.1), (3.2) and (3.3) are satisfied, provided that β is a Poisson tensor, i.e. $\beta^{[a|m} \partial_m \beta^{bc]} = 0$.

3.2 Differential geometry of a Lie algebroid

After having introduced the concept of a Lie algebroid, we now turn to the corresponding differential geometry. We will be brief here, but more details can be found in [41]. To get a general idea about the construction, let us note that the standard Riemann curvature tensor is based on the Lie bracket. Hence, a natural generalization to Lie algebroids is given by replacing the Lie bracket as $[\cdot, \cdot]_L \rightarrow [\cdot, \cdot]_E$ and inserting the anchor ρ whenever needed. This can be regarded as the main guiding principle for the following.

⁸Note that for $\xi = \xi_a dx^a$ and $\eta = \eta_a dx^a$ with $\{dx^a\}$ a basis of closed one-forms, the Koszul bracket reads explicitly $[\xi, \eta]_K = (\xi_a \beta^{ab} \partial_b \eta_m - \eta_a \beta^{ab} \partial_b \xi_m + \xi_a \eta_b \partial_m \beta^{ab}) dx^m$.

Covariant derivative

Let us start our discussion by defining a partial derivative. With $s \in \Gamma(E)$ a section of the bundle E and $f \in \mathcal{C}^\infty(M)$ a function, we define

$$D_s f = \rho(s) f. \quad (3.6)$$

For our two examples on page 15 above, this means the following:

$$\begin{aligned} E = TM : \quad D_{e_a} f &= \partial_a f & \text{where } s = e_a \text{ is a basis vector field,} \\ E = T^*M : \quad D_{e^a} f &= \beta^{am} \partial_m f & \text{where } s = e^a \text{ is a basis one-form.} \end{aligned} \quad (3.7)$$

Concerning the covariant derivative, we recall that in the usual case ∇ takes two vector fields and assigns to them a third one. This generalizes to a map $\widehat{\nabla} : \Gamma(E) \times \Gamma(E) \rightarrow \Gamma(E)$ which satisfies the following three properties

$$\begin{aligned} \widehat{\nabla}_{s_1}(s_2 + s_3) &= \widehat{\nabla}_{s_1}s_2 + \widehat{\nabla}_{s_1}s_3, \\ \widehat{\nabla}_{s_1}(f s_2) &= f \widehat{\nabla}_{s_1}(s_2) + \rho(s_1)f \cdot s_2, \\ \widehat{\nabla}_{(f s_1)}s_2 &= f \widehat{\nabla}_{s_1}s_2, \end{aligned} \quad (3.8)$$

for functions $f \in C^\infty(M)$ and section $s_i \in \Gamma(E)$. The extension to tensors of higher degree is obtained via the Leibniz rule. The action of the covariant derivative on sections $t^* \in \Gamma(E^*)$ of the dual bundle E^* is determined via the compatibility with the insertion $\langle \cdot, \cdot \rangle$.⁹ We have

$$\widehat{\nabla}_{s_1} \langle t^*, s_2 \rangle = \rho(s_1) \langle t^*, s_2 \rangle = \langle \widehat{\nabla}_{s_1} t^*, s_2 \rangle + \langle t^*, \widehat{\nabla}_{s_1} s_2 \rangle. \quad (3.9)$$

Introducing a local frame $\{\epsilon_\alpha\}$ for E and its dual $\{\epsilon^\alpha\}$, we define the Christoffel symbols by $\widehat{\Gamma}^\gamma_{\alpha\beta} = \iota_{\epsilon^\gamma} \widehat{\nabla}_{\epsilon_\alpha} \epsilon_\beta$. Using then (3.8), we can write locally

$$\widehat{\nabla}_{\epsilon^\alpha} s^\beta = D_\alpha s^\beta + \widehat{\Gamma}^\beta_{\alpha\gamma} s^\gamma \quad \text{for} \quad s = s^\alpha \epsilon_\alpha. \quad (3.10)$$

Let us emphasize that this construction is in complete analogy with the standard differential geometry calculus. We only employed a more general bundle and inserted the anchor map ρ when needed.

Curvature and torsion tensors

After having defined a covariant derivative, we can define curvature and torsion tensors. This is again in analogy to the standard case. For the curvature tensor we write

$$\widehat{R}(s_1, s_2) s_3 = [\widehat{\nabla}_{s_1}, \widehat{\nabla}_{s_2}] s_3 - \widehat{\nabla}_{[s_1, s_2]_E} s_3, \quad (3.11)$$

⁹ The insertion $\langle \cdot, \cdot \rangle : E^* \times E \rightarrow \mathbb{R}$ is characterized by $\langle \epsilon^\alpha, \epsilon_\beta \rangle = \delta^\alpha_\beta$ for $\{\epsilon_\alpha\} \in \Gamma(E)$ a basis of E and $\{\epsilon^\alpha\} \in \Gamma(E^*)$ the corresponding dual basis.

where $s_i \in \Gamma(E)$ are sections of E . Note that when replacing s_i by vector fields X, Y, Z and $[\cdot, \cdot]_E$ by the Lie bracket, we recover the familiar definition of the Riemann curvature tensor. For the torsion tensor we have similarly

$$\widehat{T}(s_1, s_2) = \widehat{\nabla}_{s_1} s_2 - \widehat{\nabla}_{s_2} s_1 - [s_1, s_2]_E. \quad (3.12)$$

To show that these expressions are indeed tensors with respect to diffeomorphisms, one has to check that they are $\mathcal{C}^\infty(M)$ -linear in all arguments. In case of, for instance, the torsion tensor, this means

$$\widehat{T}(f s_1, g s_2) = f g \widehat{T}(s_1, s_2), \quad (3.13)$$

for functions $f, g \in \mathcal{C}^\infty(M)$, which can be checked explicitly using (3.8) as well as the Leibniz property (3.1).

Metric and Levi-Civita connection

Let us finally introduce a metric g on the Lie algebroid $(E, [\cdot, \cdot]_E, \rho)$, which is an element in $\Gamma(E^* \otimes_{\text{sym}} E^*)$ assigning a number to a pair of sections $s_1, s_2 \in \Gamma(E)$. In the case of our first example on page 15 this reads

$$G(X, Y) = X^a G_{ab} Y^b, \quad (3.14)$$

for $G = G_{ab} dx^a \otimes_{\text{sym}} dx^b$ and vector fields $X = X^a \partial_a$ and $Y = Y^b \partial_b$. We require the metric g to be compatible with the connection, which means that for sections $s_i \in \Gamma(E)$

$$\widehat{\nabla}_{s_1} (g(s_2, s_3)) = g(\widehat{\nabla}_{s_1} s_2, s_3) + g(s_2, \widehat{\nabla}_{s_1} s_3). \quad (3.15)$$

If we demand in addition that the torsion tensor (3.12) vanishes, then a particular covariant derivative, the so-called Levi-Civita connection, is uniquely determined. The latter is given by the Koszul formula

$$\begin{aligned} g(\widehat{\nabla}_{s_1} s_2, s_3) = \frac{1}{2} \Big[& \rho(s_1) g(s_2, s_3) + \rho(s_2) g(s_3, s_1) - \rho(s_3) g(s_1, s_2) \\ & + g([s_1, s_2]_E, s_3) + g([s_3, s_1]_E, s_2) - g([s_2, s_3]_E, s_1) \Big]. \end{aligned} \quad (3.16)$$

In the following, the connection $\widehat{\nabla}$ is always understood to be Levi-Civita.

After having introduced the general theory, we will now give two equivalent constructions for Lie algebroids suitable for describing the field redefinitions (2.22) geometrically.

3.3 Lie algebroids on TM

In section 2.2 we have derived the field redefinitions (2.22) associated to an $O(D, D)$ transformation. Interestingly, the metric transforms by conjugation with the matrix $\gamma = d + (G - B)b$. In this section, we deduce an anchor map together with an associated bracket from γ , thus yielding a Lie algebroid for every field redefinition.

Identifying an anchor

Let us start by considering a Lie algebroid on the tangent bundle $E = TM$ of a manifold M , where the anchor map is related to the matrix γ . Recalling the submatrices a, b, c and d in a general $O(D, D)$ transformation (2.4), and keeping in mind the index structure displayed in (2.6), we have the following linear mappings

$$\begin{aligned} a : TM &\rightarrow TM, & b : T^*M &\rightarrow TM, \\ c : TM &\rightarrow T^*M, & d : T^*M &\rightarrow T^*M. \end{aligned} \quad (3.17)$$

Furthermore, the matrix $(G - B)$ can be considered as $(G - B) : TM \rightarrow T^*M$ so that we obtain

$$\gamma : T^*M \rightarrow T^*M. \quad (3.18)$$

Our aim is to identify an anchor $\rho : E \rightarrow TM$ which maps elements of the Lie algebroid bundle $E = TM$ to the tangent bundle TM . A natural candidate is (3.18), defined on the dual spaces. To determine the anchor, note that for a linear map $f : V \rightarrow W$ we have

$$\begin{aligned} f &: V \rightarrow W, \\ f^{-1} &: W \rightarrow V, \\ f^t &: W^* \rightarrow V^*, & \omega &\mapsto \omega \circ f, \\ f^* = (f^t)^{-1} &: V^* \rightarrow W^*, & \nu &\mapsto \nu \circ f^{-1}. \end{aligned} \quad (3.19)$$

Recalling (2.17), γ has to be considered as a map $E^* \rightarrow T^*M$. Therefore, the anchor $\rho : TM \rightarrow TM$ following from (3.18) is given by the inverse-transpose of γ

$$\rho = (\gamma^{-1})^t. \quad (3.20)$$

Lie algebroid bracket

Let us now determine a bracket for the Lie algebroid bundle $E = TM$. One of the main requirements on $[\cdot, \cdot]_E$ is that the anchor (3.20) is a homomorphism,

which means ρ has to satisfy equation (3.3). We start by noting that for a vector field $X = X^a e_a$ we have

$$\rho(X) = (\rho^a{}_b X^b) e_a = X^a (\rho^t{}_a)^b e_b = X^a D_a, \quad (3.21)$$

where we defined the partial derivative for the Lie algebroid as

$$D_a = (\rho^t{}_a)^b e_b. \quad (3.22)$$

In general, $\{e_a\} = \{\partial_a\}$ is a non-holonomic basis of TM which for the Lie bracket implies $[e_a, e_b]_L = f_{ab}{}^c e_c$ with $f_{ab}{}^c$ the structure constants of the underlying Lie algebra. For two vector fields $X = X^a e_a$ and $Y = Y^b e_b$ we then compute

$$[\rho(X), \rho(Y)]_L = (X^m D_m Y^a - Y^m D_m X^a + X^m Y^n F_{mn}{}^a) (\rho^t{}_a)^b e_b, \quad (3.23)$$

where we have defined

$$F_{ab}{}^c = (\rho^{-1})^c{}_m [D_a (\rho^t{}_b)^m - D_b (\rho^t{}_a)^m + (\rho^t{}_a)^p (\rho^t{}_b)^q f_{pq}{}^m]. \quad (3.24)$$

This suggests to define a new bracket $\llbracket \cdot, \cdot \rrbracket$ on $E = TM$ of the following form

$$\llbracket X, Y \rrbracket = (X^m D_m Y^a - Y^m D_m X^a + X^m Y^n F_{mn}{}^a) e_a. \quad (3.25)$$

Indeed, noting that $\rho(e_a) = (\rho^t{}_a)^b e_b$ and comparing with (3.23), we see that this bracket satisfies the homomorphism property (3.3)

$$\rho(\llbracket X, Y \rrbracket) = [\rho(X), \rho(Y)]_L. \quad (3.26)$$

Furthermore, by construction the new bracket $\llbracket \cdot, \cdot \rrbracket$ satisfies the Jacobi identity (3.2) as well as the Leibniz rule (3.1)

$$\llbracket X, fY \rrbracket = f \llbracket X, Y \rrbracket + (X^a D_a f) Y. \quad (3.27)$$

Remark

In the previous paragraph, we have shown that for every $O(D, D)$ transformation we can construct a corresponding Lie algebroid $(TM, \llbracket \cdot, \cdot \rrbracket, \rho)$ on the tangent bundle TM . However, one may argue that such a Lie algebroid could also be obtained by describing TM in a particular non-holonomic basis.

Indeed, let us define a basis $\{\tilde{e}_a\}$ of vector fields as $\tilde{e}_a = (\rho^t{}_a)^b e_b^i \partial_i$, where $\{\partial_i\} \in \Gamma(TM)$ is a holonomic basis with $[\partial_i, \partial_j]_L = 0$. For the Lie bracket in this basis we then find $[\tilde{e}_a, \tilde{e}_b]_L = F_{ab}{}^c \tilde{e}_c$, or equivalently

$$\llbracket X, Y \rrbracket = [X^a \tilde{e}_a, Y^b \tilde{e}_b]_L. \quad (3.28)$$

Or in other words, the Lie algebroid bracket $\llbracket \cdot, \cdot \rrbracket$ is just the ordinary Lie bracket in the basis $\{\tilde{e}_a\}$. Therefore, for any anchor $\rho = (\gamma^t)^{-1}$ we could choose a corresponding diffeomorphism $\gamma = (\mathbf{A}^t)^{-1}$ which gives rise to a Lie algebroid bracket. In the case of geometric transformations $\mathcal{M} \in G_{\text{geom}}$, this is the expected form, but for β -transforms (2.14) with $\gamma = 1 + (G - B)\beta$ the corresponding diffeomorphism $(\mathbf{A}^t)^{-1} = 1 + (G - B)\beta$ involves the dynamical fields themselves. This is not what one usually understands by a diffeomorphism in differential geometry, and must rather be considered as a *generalized* change of coordinates.

These observations can be summarized by saying that β -transforms go beyond the usual notions of differential geometry, and the Lie algebroid presented in this section provides the appropriate mathematical framework to describe both geometric transformations G_{geom} and non-geometric β -transforms.

3.4 Lie algebroids on T^*M

After having constructed a Lie algebroid on TM , we next investigate how a Lie algebroid structure can be defined on the cotangent bundle T^*M . For our second example in section 2.3, such a Lie algebroid was constructed in [36].

Construction

Let us note that the metric G on the manifold M can be seen as a linear mapping $G : TM \rightarrow T^*M$, while the inverse gives a map $G^{-1} : T^*M \rightarrow TM$. Combining this observation with (3.18), we arrive at the following picture

$$\begin{array}{ccc} E_2 = T^*M & \xrightarrow{\gamma} & T^*M \\ \hat{G}^{-1} \downarrow & & \downarrow G^{-1} \\ E_1 = TM & \xrightarrow{\rho = (\gamma^t)^{-1}} & TM \end{array} \quad (3.29)$$

where on the left-hand side we have the Lie algebroid bundles $E_1 = TM$ and $E_2 = T^*M$, while on the right-hand side there are the standard tangent and cotangent bundles of the manifold. An anchor for a Lie algebroid on T^*M can therefore be defined as follows

$$\tilde{\rho} = G^{-1} \circ \gamma : T^*M \rightarrow TM. \quad (3.30)$$

For a one-form $\xi = \xi_\alpha e^\alpha$, locally the anchor $\tilde{\rho}$ acts as follows

$$\tilde{\rho}(\xi) = (\tilde{\rho}^{b\alpha} \xi_\alpha) e_b = \xi_\alpha (\tilde{\rho}^t)^{\alpha b} e_b = \xi_\alpha (\gamma^t)^\alpha_\beta G^{\beta c} e_c, \quad (3.31)$$

where we denote indices related to T^*M by Greek letters. Analogous to the bracket (3.25) on TM , we can define a bracket on T^*M as

$$\llbracket \xi, \eta \rrbracket_* = (\xi_\mu D^\mu \eta_\alpha - \eta_\mu D^\mu \xi_\alpha + \xi_\mu \eta_\nu Q_\alpha^{\mu\nu}) e^\alpha, \quad (3.32)$$

with the associated partial derivative given by

$$D^\alpha = (\tilde{\rho}^t)^{\alpha m} e_m, \quad (3.33)$$

and structure constants of the form

$$Q_\alpha^{\beta\gamma} = (\tilde{\rho}^{-1})_{\alpha m} [D^\beta (\tilde{\rho}^t)^{\gamma m} - D^\gamma (\tilde{\rho}^t)^{\beta m} + (\tilde{\rho}^t)^{\beta p} (\tilde{\rho}^t)^{\gamma q} f_{pq}^m]. \quad (3.34)$$

Again, one can verify that (3.32) satisfies the homomorphism property (3.3) as well as the corresponding Leibniz rule and Jacobi identity. Therefore, we obtain a Lie algebroid $(T^*M, \llbracket \cdot, \cdot \rrbracket, \tilde{\rho})$ on the cotangent bundle.

Remarks

Let us close this subsection with two remarks:

- For an antisymmetric anchor with an appropriate Poisson condition, the bracket (3.32) coincides with the corresponding Koszul bracket shown in equation (3.5) (cf. [4]). This is the realm of Poisson geometry. However, (3.32) is more general in the sense that it is also valid for the symmetric part of an anchor.
- The bracket (3.32) on the cotangent bundle T^*M can be related to the bracket (3.25) on TM via

$$\llbracket \xi, \eta \rrbracket_* = \widehat{G} \left(\llbracket \widehat{G}^{-1} \xi, \widehat{G}^{-1} \eta \rrbracket \right), \quad (3.35)$$

where \widehat{G} is the transformed metric (2.17). Thus, with the metric only the indices are raised and lowered, which means that the differential geometry constructed on $(T^*M, \llbracket \cdot, \cdot \rrbracket_*, \tilde{\rho})$ is equivalent to the one constructed on $(TM, \llbracket \cdot, \cdot \rrbracket, \rho)$.

3.5 Examples

Let us illustrate the above constructions within the two frames mentioned in section 2.3. More concretely, we determine explicitly the Lie algebroids corresponding to the $O(D, D)$ transformations (2.27) and (2.34).

Frame I

Inserting the $O(D, D)$ transformation (2.27) into the map (2.18) yields the matrix

$$\gamma_I = (\mathbb{1} + BG^{-1})^{-1}. \quad (3.36)$$

Together with (2.24), we can then confirm that the general formulas (2.17) and (2.20) reproduce the field redefinition (2.29). The anchor (3.20) is given by

$$\rho_I = \mathbb{1} - G^{-1}B, \quad (3.37)$$

and the corresponding structure constants of the Lie algebroid bracket $[\![\cdot, \cdot]\!]_I$ can be computed from (3.24). In particular, we find

$$(F_I)_{ab}{}^c = 2 \left[(G + B)G^{-1} \right]_{[a}{}^m \partial_m \left[BG^{-1} \right]_{b]}{}^n \left[G(G + B)^{-1} \right]_n{}^c, \quad (3.38)$$

where for simplicity we have set to zero the structure constants $f_{ab}{}^c$ of the coordinate indices. Working out in detail (3.38) results in a rather lengthy expression which we do not present here. However, the above information completely characterizes the Lie algebroid $(TM, [\![\cdot, \cdot]\!]_I, \rho_I)$.

Frame II

The transformation (2.34) can be used to provide an example of a Lie algebroid on the cotangent bundle. From the $O(D, D)$ transformation we can read off the map

$$\gamma_{II} = -\mathbb{1} - (G - B)\hat{\beta} = -G\hat{\beta}, \quad (3.39)$$

where we employed $\hat{\beta} = B^{-1}$. Again, using (2.17) and (2.20) we can confirm the redefinition (2.36). In addition, invoking (3.20) and (3.30) we obtain the corresponding anchor on TM and T^*M as

$$\rho_{II} = G^{-1}\hat{\beta}^{-1} \quad \text{and} \quad \tilde{\rho}_{II} = -\hat{\beta}, \quad (3.40)$$

respectively. The structure constants (3.24) and (3.34) of the Lie algebroid brackets on TM and T^*M are computed as follows,

$$\begin{aligned} (F_{II})_{ab}{}^c &= 2 \left[\hat{\beta}^{-1}G^{-1} \right]_{[a}{}^m \partial_m \left[\hat{\beta}^{-1}G^{-1} \right]_{b]}{}^n \left[G\hat{\beta} \right]_n{}^c, \\ (Q_{II})_\alpha{}^{\beta\gamma} &= 2 \hat{\beta}^{[\beta|m} \partial_m \hat{\beta}^{|\gamma]n} \left[\hat{\beta}^{-1} \right]_{n\alpha}, \end{aligned} \quad (3.41)$$

where for simplicity we set $f_{ab}{}^c$ to zero. Note that the structure constants Q_{II} take a particular simple form for this example and match with the corresponding expression in [36]. Furthermore, in view of our observations at the end of section 3.4, the anchor $\tilde{\rho}_{II}$ is interesting as it is antisymmetric. If we require $\hat{\beta}$ to satisfy the *quasi-Poisson* condition

$$\hat{\beta}^{am} \partial_m \hat{\beta}^{bc} + \text{cycl.} = -\hat{\beta}^{am} \hat{\beta}^{bn} \hat{\beta}^{ck} H_{mnk} \quad (3.42)$$

for a three-form $H = \frac{1}{3!} H_{\alpha\beta\gamma} e^\alpha \wedge e^\beta \wedge e^\gamma$, the bracket (3.32) coincides with the so-called H -twisted Koszul bracket. Indeed, we find

$$[\![\xi, \eta]\!]_{II} = L_{\hat{\beta}(\xi)}\eta - \iota_{\hat{\beta}(\eta)}d\xi + \iota_{\hat{\beta}(\eta)}\iota_{\hat{\beta}(\xi)}H = [\xi, \eta]_K^H, \quad (3.43)$$

where as before $\hat{\beta}(\xi) = \xi_\alpha \hat{\beta}^{\alpha b} e_b$ and $L_X = \iota_X \circ d + d \circ \iota_X$. This provides the connection to [36] where this particular Lie algebroid $(T^*M, [\cdot, \cdot]_K^H, \hat{\beta})$ has been studied in detail.

4 Differential geometry in non-geometric frames

In this section, we establish a connection between the differential geometry of a Lie algebroid on E , on the one hand, and the standard geometry on TM , on the other hand. In particular, utilizing the field redefinitions (2.22), we derive a correspondence between the respective differential geometric objects. This provides a general framework for the formulation of gravity theories which are related to standard gravity via $O(D, D)$ transformations.

Our setup is as follows: we start from a general Lie algebroid $(E, [\cdot, \cdot]_E, \rho)$ equipped with a metric $g \in \Gamma(E^* \otimes_{\text{symm}} E^*)$ and for which the anchor $\rho : E \rightarrow TM$ is invertible. For our previous example of a Lie algebroid on TM one has $g = \hat{G}$. Moreover, we assume this metric to be related to the Riemannian metric G by applying the anchor as follows

$$G = (\otimes^2 \rho^*)(g) \quad \Longleftrightarrow \quad g = (\otimes^2 \rho^t)(G), \quad (4.1)$$

where $\rho^* : E^* \rightarrow T^*M$ is the dual anchor and $\rho^t : T^*M \rightarrow E^*$ the transpose anchor, cf. (3.19). The relation (4.1) contains the redefinition discussed above as it is in accordance with (2.17) for $\rho = (\gamma^{-1})^t$.

4.1 Relating Riemannian geometry to non-geometry

In this section we work out in detail the relation between the differential geometric objects appearing for the Lie algebroid and the familiar ones from standard Riemannian geometry.¹⁰ Let $\{e_a\}$ and $\{\epsilon_\alpha\}$ be a local frame for TM and E , respectively. Using the corresponding dual bases, we can write the metrics as $G = G_{ab} e^a \otimes e^b$ and $g = g_{\alpha\beta} \epsilon^\alpha \otimes \epsilon^\beta$. Thus, the field redefinition (4.1) in local coordinates reads¹¹

$$G_{ab} = (\rho^*)_a{}^\alpha (\rho^*)_b{}^\beta g_{\alpha\beta}. \quad (4.3)$$

¹⁰The reader not interested in the mathematical details may go directly to page 26, where a summary of all relevant formulas of this subsection can be found.

¹¹The conventions for the indices are as follows

$$\rho \equiv \rho^a{}_\alpha, \quad \rho^{-1} \equiv (\rho^{-1})^\alpha{}_a, \quad \rho^t \equiv (\rho^t)_\alpha{}^a, \quad \rho^* \equiv (\rho^*)_a{}^\alpha. \quad (4.2)$$

Note that here the index α of ρ , i.e. the one corresponding to the Lie algebroid, is chosen to be downstairs. However, when discussing particular examples, for instance $E = T^*M$, it might be more convenient to change the index structure to $\rho^{a\beta}$.

In a coordinate-free notation, for sections $s, t \in \Gamma(E)$ one can equivalently write

$$G(\rho(s), \rho(t)) = g(s, t), \quad (4.4)$$

as $\rho^* = (\rho^t)^{-1}$ and for a one-form $\xi \in T^*M$ one has

$$\rho^t(\xi)(s) = (\xi_a (\rho^t)_\alpha{}^a) s^\alpha = \xi_a (\rho^a{}_\alpha s^\alpha) = \xi(\rho(s)). \quad (4.5)$$

In the following, sections of E are denoted by s, t and dual sections by s^*, t^* .

The connections

Let us turn to the Levi-Civita connection on the Lie algebroid E . Denoting the standard Levi-Civita connection on TM by ∇ and employing (4.4) in the Koszul formula (3.16) together with the anchor property (3.3), we find

$$\begin{aligned} G(\rho(\widehat{\nabla}_r s), \rho(t)) &= g(\widehat{\nabla}_r s, t) \\ &= G(\nabla_{\rho(r)} \rho(s), \rho(t)). \end{aligned} \quad (4.6)$$

Thus, by non-degeneracy of the metrics we infer

$$\rho(\widehat{\nabla}_s t) = \nabla_{\rho(s)} \rho(t), \quad \rho^*(\widehat{\nabla}_s t^*) = \nabla_{\rho(s)} \rho^*(t^*). \quad (4.7)$$

The second identity follows from compatibility with the insertion and the first identity. This can be seen as follows. First observe that $\langle s, t^* \rangle = \langle \rho(s), \rho^*(t^*) \rangle$. In view of the compatibility of $\widehat{\nabla}$ and ∇ with the insertion, this implies

$$\begin{aligned} \langle \widehat{\nabla}_r s, t^* \rangle + \langle s, \widehat{\nabla}_r t^* \rangle &= \langle \nabla_{\rho(r)} \rho(s), \rho^*(t^*) \rangle + \langle \rho(s), \nabla_{\rho(r)} \rho^*(t^*) \rangle \\ &= \langle \widehat{\nabla}_r s, t^* \rangle + \langle \rho(s), \nabla_{\rho(r)} \rho^*(t^*) \rangle \\ &= \langle \widehat{\nabla}_r s, t^* \rangle + \langle s, \rho^t(\nabla_{\rho(r)} \rho^*(t^*)) \rangle. \end{aligned} \quad (4.8)$$

We therefore have

$$\widehat{\nabla}_r t^* = \rho^t(\nabla_{\rho(r)} \rho^*(t^*)) \iff \rho^*(\widehat{\nabla}_s t^*) = \nabla_{\rho(s)} \rho^*(t^*), \quad (4.9)$$

and so (4.7) establishes the connection between the Levi-Civita connections in both frames. The corresponding connection coefficients in local coordinates are defined in the standard way

$$\Gamma_{ab}^c = \iota_{e^c} \nabla_{e_a} e_b, \quad \widehat{\Gamma}_{\alpha\beta}^\gamma = \iota_{e^\gamma} \widehat{\nabla}_{e_\alpha} e_\beta. \quad (4.10)$$

Using then (4.7), the relation between the Christoffel symbols in the Riemannian and Lie algebroid setting reads

$$\widehat{\Gamma}_{\alpha\beta}^\gamma = (\rho^{-1})^\gamma{}_c \rho^a{}_\alpha \rho^b{}_\beta \Gamma_{ab}^c + (\rho^{-1})^\gamma{}_b \rho^a{}_\alpha \partial_a \rho^b{}_\beta. \quad (4.11)$$

Torsion and curvature

The relation (4.7) found above is of the same type as the relation between the brackets given by the anchor property, i.e. $\rho([s, t]_E) = [\rho(s), \rho(t)]_L$. Since the torsion and the curvature are defined in terms of the connection and the bracket (cf. (3.11) and (3.12)), we can relate them accordingly. Thus, by applying (4.7) and the anchor property (3.3) we obtain

$$\widehat{T}(s, t) = \rho^{-1}(T(\rho(s), \rho(t))), \quad \widehat{R}(s, t)r = \rho^{-1}(R(\rho(s), \rho(t))\rho(r)), \quad (4.12)$$

where T and R denote the torsion and curvature with respect to ∇ on TM . In a local frame, the relation between the curvatures reads

$$\widehat{R}^\alpha_{\beta\gamma\delta} = \langle \epsilon^\alpha, \widehat{R}(\epsilon_\gamma, \epsilon_\delta)\epsilon_\beta \rangle = (\rho^{-1})^\alpha_a \rho^b_\beta \rho^c_\gamma \rho^d_\delta R^a_{bcd}, \quad (4.13)$$

which is simply the contraction of all indices of the Riemann tensor R^a_{bcd} with the anchor. For the Ricci tensor and Ricci scalar this implies

$$\widehat{R}_{\alpha\beta} = \widehat{R}^\gamma_{\alpha\gamma\beta} = \rho^a_\alpha \rho^b_\beta R_{ab}, \quad \widehat{R} = g^{\alpha\beta} \widehat{R}_{\alpha\beta} = G^{ab} R_{ab} = R, \quad (4.14)$$

where we employed (4.3) for the Ricci scalar. Let us remark that for a covariant theory, all terms appearing in the corresponding Lagrangian must be scalars. From (4.13) and (4.3) we then infer that all scalars built from curvature tensors are equal, e.g. $\widehat{R}_{\alpha\beta} \widehat{R}^{\alpha\beta} = R_{ab} R^{ab}$.

The exterior derivative

As was done for the connection, also the exterior derivative can be transferred to the Lie algebroid by applying the anchor. Indeed, any Lie algebroid can be equipped with a nilpotent exterior derivative as follows

$$\begin{aligned} d_E \theta^*(s_0, \dots, s_n) &= \sum_{i=0}^n (-1)^i \rho(s_i) \theta^*(s_0, \dots, \hat{s}_i, \dots, s_n) \\ &\quad + \sum_{i < j} (-1)^{i+j} \theta^*([s_i, s_j]_E, s_0, \dots, \hat{s}_i, \dots, \hat{s}_j, \dots, s_n), \end{aligned} \quad (4.15)$$

where $\theta^* \in \Gamma(\Lambda^n E^*)$ is the analog of an n -form on the Lie algebroid and \hat{s}_i indicates the omission of that entry. The Jacobi identity of the bracket $[\cdot, \cdot]_E$ implies that (4.15) satisfies $(d_E)^2 = 0$. The anchor property and the corresponding formula for the de Rahm differential allow to compute

$$\begin{aligned} ((\Lambda^{n+1} \rho^*)(d_E \theta^*))(X_0, \dots, X_n) &= (d_E \theta^*)(\rho^{-1}(X_0), \dots, \rho^{-1}(X_n)) \\ &= d((\Lambda^n \rho^*)(\theta^*))(X_0, \dots, X_n) \end{aligned} \quad (4.16)$$

for sections $X_i \in \Gamma(TM)$. The relation (4.16) describes how exact terms translate in general. As an example, for the partial derivative ($n = 0$) this locally gives

$$D_\alpha = \rho(\epsilon_\alpha) = \rho^a{}_\alpha \partial_a. \quad (4.17)$$

We will come back to this in the next section, when we discuss the effect of the field redefinition on the H -flux.

Summary

We now summarize the relevant formulas connecting the differential geometric quantities of the Lie algebroid E to the standard geometric framework on the tangent space TM :

metric	$g_{\alpha\beta} = \rho^a{}_\alpha \rho^b{}_\beta G_{ab},$	
LC connection	$\widehat{\Gamma}^\gamma{}_{\alpha\beta} = (\rho^{-1})^\gamma{}_c \rho^a{}_\alpha \rho^b{}_\beta \Gamma^c{}_{ab} + (\rho^{-1})^\gamma{}_b \rho^a{}_\alpha \partial_a \rho^b{}_\beta,$	
curvature tensor	$\widehat{R}^\alpha{}_{\beta\gamma\delta} = (\rho^{-1})^\alpha{}_a \rho^b{}_\beta \rho^c{}_\gamma \rho^d{}_\delta R^a{}_{bcd},$	(4.18)
Ricci tensor	$\widehat{R}_{\alpha\beta} = \rho^a{}_\alpha \rho^b{}_\beta R_{ab},$	
Ricci scalar	$\widehat{R} = R,$	
partial derivative	$D_\alpha = \rho^a{}_\alpha \partial_a.$	

As one can see, except for the coefficients of the Levi-Civita connection all the expressions are related simply by applying the anchor map ρ .

4.2 Gauge transformations

The objects discussed so far behave as tensors under coordinate changes, cf. section 3.2. However, applying the anchor generically imposes a dependence on the B -field upon the redefined objects. For this reason and for covering all the symmetries of the string action (2.38), we have to study how gauge transformations translate under a field redefinition.

We consider the redefinition of the standard Kalb-Ramond field B

$$B = (\Lambda^2 \rho^*)(\mathfrak{b}), \quad (4.19)$$

with $\mathfrak{b} \in \Gamma(\Lambda^2 E^*)$, which in local coordinates reads

$$B_{ab} = (\rho^*)^\alpha{}_a (\rho^*)^\beta{}_b \mathfrak{b}_{\alpha\beta}. \quad (4.20)$$

Note that for our case of interest, namely the field redefinition (2.23), the \mathfrak{b} -field takes the form

$$\mathfrak{b} = \hat{\gamma} \hat{\delta}^t - \widehat{G}, \quad (4.21)$$

where the matrices $\hat{\gamma}$ and $\hat{\delta}$ generically depend on \hat{G} and \hat{B} . The gauge transformations for B read

$$B \rightarrow B + d\xi \quad (4.22)$$

with ξ denoting a one-form. Let us stress that, since the anchor generically depends on G and B , any object containing the anchor transforms under gauge transformations. Thus, we have to carefully distinguish objects whose gauge dependence just stems from the anchor from those that are inherently gauge dependent.

Due to the inherent B dependence of \mathfrak{b} (4.20), its overall variation $\delta_\xi \mathfrak{b}$ under gauge transformations receives a contribution according to

$$(\Lambda^2 \rho^*)(\mathfrak{b} + \hat{\delta}_\xi \mathfrak{b}) = B + d\xi. \quad (4.23)$$

Inverting (4.16), we find

$$\hat{\delta}_\xi \mathfrak{b} = (\Lambda^2 \rho^t)(d\xi) = d_E(\rho^t \xi). \quad (4.24)$$

A second contribution comes from the possible B -dependence of the anchor so that overall we get

$$\delta_\xi \mathfrak{b} = \Delta_\xi^2(B) + \hat{\delta}_\xi \mathfrak{b}. \quad (4.25)$$

Here we introduced the variation of the anchor as $\Delta_\xi^n = \delta_\xi(\otimes^n \rho^t)$. Since the metric G is gauge invariant, a non-trivial gauge variation of g can only arise via the anchor so that

$$\delta_\xi g = \delta_\xi((\otimes^2 \rho^t)(G)) = \Delta_\xi^2(G) + (\otimes^2 \rho^t)(\delta_\xi G) = \Delta_\xi^2(G). \quad (4.26)$$

We want to implement the appearance of non-trivial gauge variations related to the B -dependence of the anchor in a consistent modified tensor calculus. For this purpose we will introduce the notion of ρ -tensors.¹² In particular, we require the metric to be a ρ -tensor. This suggests to define such a tensor by its relation to a gauge invariant object on TM . More precisely, we make the following definition:

Definition: A section $\tau \in \Gamma((\otimes^r E) \otimes (\otimes^s E^*))$ of the Lie algebroid E is called a ρ -tensor if

$$[(\otimes^r \rho) \otimes (\otimes^s \rho^*)](\tau) \in \Gamma((\otimes^r TM) \otimes (\otimes^s T^*M)) \quad (4.27)$$

is gauge invariant. A ρ -gauge transformation of an n -form $\tau \in \Gamma(\Lambda^n E^*)$ is characterized by an $(n-1)$ -form $\mathfrak{a} \in \Gamma(\Lambda^{n-1} E^*)$ as

$$\tau \rightarrow \tau + d_E \mathfrak{a}. \quad (4.28)$$

¹²Note that this generalizes the concept of β -tensors introduced for the specific non-geometric frame studied in [35, 36].

In other words, this definition characterizes a ρ -tensor τ as an object whose image under the anchor map is invariant under B -field gauge transformation. Written in components, a section $\tau^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s}$ is a ρ -tensor if there exists a gauge invariant standard (r, s) -tensor T with

$$T^{a_1 \dots a_r}_{b_1 \dots b_s} = \rho^{a_1}_{\alpha_1} \dots \rho^{a_r}_{\alpha_r} (\rho^*)_{b_1}^{\beta_1} \dots (\rho^*)_{b_s}^{\beta_s} \tau^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s}. \quad (4.29)$$

Note that the differential geometry we constructed above gives ρ -tensors right away. Indeed, (4.7) and the anchor property written as

$$\widehat{\nabla}_s t = \rho^{-1}(\nabla_{\rho(s)} \rho(t)), \quad [s, t]_E = \rho^{-1}([\rho(s), \rho(t)]_L) \quad (4.30)$$

imply that the covariant derivative as well as the Lie algebroid bracket respect the tensoriality. Equation (4.25) shows that \mathfrak{b} is not a ρ -tensor but receives a defect $\widehat{\delta}_\xi \mathfrak{b}$, which is related to its inherent gauge dependence.

Since the algebroid differential (4.15) is nilpotent, the natural ρ -gauge invariant object built from \mathfrak{b} is

$$\Theta = d_E \mathfrak{b} \in \Gamma(\Lambda^3 E^*). \quad (4.31)$$

Using (4.15), locally this can be written as¹³

$$\Theta_{\alpha\beta\gamma} = \widehat{\nabla}_{[\alpha} \mathfrak{b}_{\beta\gamma]}, \quad (4.32)$$

where we abbreviated $\widehat{\nabla}_{\epsilon_\alpha} \equiv \widehat{\nabla}_\alpha$. As a consequence, the Bianchi identity

$$d_E \Theta = 0 \iff \widehat{\nabla}_{[\alpha} \Theta_{\beta\gamma\delta]} = 0 \quad (4.33)$$

is satisfied. Moreover, using (4.16) we obtain

$$\Theta = d_E((\Lambda^2 \rho^t)B) = (\Lambda^3 \rho^t)dB = (\Lambda^3 \rho^t)H, \quad (4.34)$$

i.e. Θ is precisely the redefinition of the B -gauge invariant field H . This also confirms that, unlike \mathfrak{b} , Θ is a ρ -tensor.

Remark

Motivated by the examples appearing in the literature [33, 34], one might also want a transformation relating the two-form B to a two-vector $\beta \in \Gamma(\Lambda^2 E)$. This is different from (4.20) where $\mathfrak{b} \in \Gamma(\Lambda^2 E^*)$, and requires a map $\sigma : E \rightarrow T^*M$. This is apparently not the anchor, but recalling (3.29), we do have two natural candidates for such a map:

$$\begin{aligned} \sigma_1 : E &\rightarrow T^*M; s \mapsto \rho^* \circ g^\sharp(s) \\ \sigma_2 : E &\rightarrow T^*M; s \mapsto G^\sharp \circ \rho(s). \end{aligned} \quad (4.35)$$

¹³Note that the symmetric part of the connection drops out due to the antisymmetrization.

Using the translation of the metrics (4.1) yields $\sigma_1 = \sigma_2 \equiv \sigma$. Then, the redefinition reads

$$B = (\Lambda^2 \sigma) \beta = (\Lambda^2 (\rho^* \circ g^\sharp)) \beta = (\Lambda^2 \rho^*) ((\Lambda^2 g^\sharp) \beta). \quad (4.36)$$

Hence also this case fits into the general picture (4.20) by identifying $\mathfrak{b} = (\Lambda^2 g^\sharp) \beta$. This was already used in (2.30).

4.3 The general redefined action

In the previous sections, we discussed all relevant ingredients for giving the general action arising from the NS-NS Lagrangian (2.38) by redefining the metric and the B -field according to (4.1) and (4.20). For the new action, we have to give the new Ricci scalar, the flux term and the dilaton term. Moreover, also the measure changes. Let us start by discussing the latter.

The standard measure behaves under a field redefinition (4.1) as follows:

$$\sqrt{|G|} = \sqrt{|((\rho^*)^\alpha_a (\rho^*)^\beta_b g_{\alpha\beta})|} = \sqrt{|g|} |\rho^*|. \quad (4.37)$$

The measure is well-defined by recalling that the anchor is invertible.

The remaining terms in the action have been discussed in (4.13) for the curvature and in (4.34) for the flux term. Hence the translation of the Ricci scalar is straightforward. For the H -flux term we observe that by (4.1) and (4.34)

$$H_{abc} G^{am} G^{bn} G^{ck} H_{mnk} = \Theta_{\alpha\beta\gamma} g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} \Theta_{\mu\nu\rho}, \quad (4.38)$$

where Θ has been defined in (4.32). Using (4.17), the dilaton term translates analogously

$$\partial_a \phi G^{ab} \partial_b \phi = D_\alpha \phi g^{\alpha\beta} D_\beta \phi, \quad (4.39)$$

where the dilaton itself does not transform. Note that ρ -scalars are related to usual scalars without any contraction with the anchor. As every term in the Lagrangian is a scalar, each individual term maps to the corresponding ρ -scalar directly. Putting all these pieces together, we obtain from the NS-NS action (2.38) the final action in a non-geometric frame

$$\mathcal{S} = -\frac{1}{2\kappa^2} \int d^n x \sqrt{|g|} |\rho^*| e^{-2\phi} \left(\widehat{R} - \frac{1}{12} \Theta_{\alpha\beta\gamma} \Theta^{\alpha\beta\gamma} + 4 D_\alpha \phi D^\alpha \phi \right). \quad (4.40)$$

By construction, this action is invariant under diffeomorphisms and two-form gauge transformations, whose inherent part acts like a \mathfrak{b} -gauge transformation

$$\mathfrak{b} \rightarrow \mathfrak{b} + d_E \mathfrak{a} \quad (4.41)$$

for $\mathfrak{a} \in \Gamma(E^*)$. Hence (4.40) bears the redefined analogs of the symmetries of the geometric action (2.38) and provides the generalization of the action (2.39) to any non-geometric frame.

Let us emphasize that by construction (4.40) and (2.38) are directly related by the field redefinition (4.1) and (4.20):

$$\mathcal{S}(g, \mathfrak{b}) \xleftrightarrow[\mathfrak{b}=(\wedge^2 \rho^t)B]{g=(\otimes^2 \rho^t)G} S(G, B). \quad (4.42)$$

String action in Frame I

Let us recall that the anchor in frame I (2.29) was given in eq. (3.37) as

$$\rho_1 = \mathbb{1} - G^{-1}B = \mathbb{1} + \widehat{\beta}\widehat{G}, \quad (4.43)$$

where for the last step we employed the field redefinitions (2.30) and (2.32). The partial derivative (4.17) of the Lie algebroid then becomes

$$D_a = \partial_a - \widehat{G}_{a\beta}\widehat{\beta}^{\beta c}\partial_c. \quad (4.44)$$

For the measure of the redefined action, we can use the relation derived in equation (4.37), which leads to

$$\sqrt{|\widehat{G}^{-1}|}|\widehat{G}^{-1} - \widehat{\beta}|^{-1}. \quad (4.45)$$

The components of the flux Θ can be determined for instance from equation (4.34) by recalling that in our conventions $H_{abc} = 3\partial_{[a}B_{bc]}$. We then compute

$$\Theta_{\alpha\beta\gamma} = 3(\mathbb{1} - \tilde{G}\widehat{\beta})_{[\alpha}{}^a(\mathbb{1} - \widehat{G}\widehat{\beta})_{\beta}{}^b(\mathbb{1} - \widehat{G}\widehat{\beta})_{\gamma]}{}^c\partial_a(\widehat{G}_{bm}\widehat{\beta}^{mn}\widehat{G}_{nc}). \quad (4.46)$$

As one can see, the non-geometric analog of the H -flux is a rather complicated expression. However, the flux (4.46) does contain the familiar R -flux term $R^{abc} = 3\widehat{\beta}^{[a|m}\partial_m\widehat{\beta}^{bc]}$, which is accompanied by a plenitude of additional terms

$$\Theta_{\alpha\beta\gamma} = -3\widehat{G}_{\alpha a}\widehat{G}_{\beta b}\widehat{G}_{\gamma c}[\widehat{\beta}^{[a|m}\partial_m\widehat{\beta}^{bc]}] + \mathcal{O}(\partial\widehat{G}) + \mathcal{O}(\partial\widehat{\beta}). \quad (4.47)$$

When expressing the Ricci scalar in terms of the fields \widehat{G} and $\widehat{\beta}$, we obtain similarly involved expressions, and we refrain from presenting them here. The explicit form of the action in the $(\widehat{G}, \widehat{\beta})$ -frame, modulo total-derivative terms, can be found in [32, 33, 34].

String action in Frame II

For our second example we recall that the field redefinition was given in (2.36). Furthermore, the corresponding anchor $\tilde{\rho}_{\text{II}}$ for a Lie algebroid on T^*M has been derived in (3.40)

$$\tilde{\rho}_{\text{II}} = -\hat{\beta}, \quad (4.48)$$

where $\hat{\beta}$ is an antisymmetric bi-vector. The associated partial derivative can be determined as

$$D^\alpha = \hat{\beta}^{\alpha b} \partial_b. \quad (4.49)$$

The measure for the action in the redefined field variables can be inferred for instance from (2.36) and takes the form

$$\sqrt{|\hat{g}|} |\hat{\beta}|^{-1}. \quad (4.50)$$

For the flux Θ we employ again the relation shown in (4.34) which, using (2.36), allows us to write

$$\Theta^{\alpha\beta\gamma} = -3\hat{\beta}^{[\alpha|m}\partial_m\hat{\beta}^{|\beta\gamma]}. \quad (4.51)$$

The curvature scalar can be constructed along the lines outlined above, as was done in [35, 36]. Using these building blocks in (4.40), one can construct the action (2.39) in the non-geometric $(\hat{g}, \hat{\beta})$ -frame [35, 36].

5 Further aspects of non-geometric gravity

In this section we discuss a couple of interesting aspects of the generalized gravity action (4.40) in the non-geometric frame. First, we will discuss how it fits into the formalism of double field theory (DFT), which is a candidate to provide a unified framework for the geometric and non-geometric phases of (bosonic) string theory, at least at tree-level. Second, we apply the formalism developed above to perform the translation of the remaining terms in the superstring action. This includes terms from the Ramond-Ramond sector as well as fermionic terms. We also comment on higher order α' -corrections and the tree-level equations of motion of the action (4.40). Finally, we discuss the important question, in which sense non-geometric frames are appropriate or useful to describe non-geometric backgrounds.

5.1 Relation to double field theory

The goal of this section is whether and how the action in the non-geometric frames (4.40) does arise in DFT. In the following, we describe how this works for the case of rigid $O(D, D)$ transformations. Non-geometric frames related to $O(D, D)$ -transformations which contain generic local β -transformations go beyond the regime of the DFT action and involve field redefinitions which cannot be generated by its symmetries.

Basics of DFT

In DFT not only the dimension of the bundle is doubled but even the number of coordinates. This is done by also introducing the canonical conjugate variables for the string winding operators, which are called winding coordinates \tilde{x}_i , and arranging them into a doubled vector $X^M = (\tilde{x}_i, x^i)$.

As was developed in [10, 7, 8, 9], one can formulate an action on this doubled space in which the generalized metric appears explicitly¹⁴

$$S_{\text{DFT}} = -\frac{1}{2\kappa^2} \int d^D x d^D \tilde{x} e^{-2d} \left(\frac{1}{8} \mathcal{H}^{MN} (\partial_M \mathcal{H}^{KL}) (\partial_N \mathcal{H}_{KL}) \right. \\ \left. - \frac{1}{2} \mathcal{H}^{MN} (\partial_N \mathcal{H}^{KL}) (\partial_L \mathcal{H}_{MK}) - 2 (\partial_M d) (\partial_N \mathcal{H}^{MN}) + 4 \mathcal{H}^{MN} (\partial_M d) (\partial_N d) \right). \quad (5.1)$$

Note that here $\partial_M = (\tilde{\partial}^i, \partial_i)$, and d denotes the dilaton which is defined as $\exp(-2d) = \sqrt{|G|} \exp(-2\phi)$. This action has been determined by invoking a number of symmetries: First it was required to be invariant under local diffeomorphisms of the coordinates X^M , i.e. $(\tilde{x}_i, x^i) \rightarrow (\tilde{x}_i + \tilde{\xi}_i(X), x^i + \xi^i(X))$ ¹⁵. Second, the action is invariant under a global or rigid $O(D, D)$ symmetry, which acts as

$$\begin{aligned} \mathcal{H}' &= h^t \mathcal{H} h, & d' &= d, \\ X' &= h X, & \partial' &= (h^t)^{-1} \partial, \end{aligned} \quad (5.2)$$

with¹⁶

$$h = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{pmatrix}. \quad (5.3)$$

¹⁴Note that usually, one splits the n -dimensional space-time into a D -dimensional compact part, and an $(n - D)$ -dimensional non-compact part. The doubling of coordinates takes place only in the compact space, the action for the other coordinates is unchanged. This is implicitly assumed in (5.1).

¹⁵The x^i dependence of these two diffeomorphisms include both standard diffeomorphisms and B -field gauge transformations. Note that the winding coordinate dependence of ξ^i also gives what one might call β -field gauge transformations.

¹⁶The constant matrix \mathfrak{b} should not be confused with the space-time dependent field introduced in (4.19).

For this manifest $O(D, D)$ invariance this action has to be supplemented by the so-called strong constraint

$$\partial_i A \tilde{\partial}^i B + \tilde{\partial}^i A \partial_i B = 0 , \quad (5.4)$$

with A, B arbitrary fields. Whether this constraint can be weakened in compactifications of DFT has recently been analyzed in [42]. Solving (5.4) by setting to zero the derivative with respect to the winding coordinates $\tilde{\partial}^i = 0$, the double field theory action reduces to the action in the geometric frame (2.38).

Let us also recall an alternative formulation of DFT. One introduces the $O(D, D)$ covariant partial derivatives

$$\begin{aligned} \mathcal{D}_i &= \partial_i - \mathcal{E}_{ik} \tilde{\partial}^k , \\ \overline{\mathcal{D}}_i &= \partial_i + \mathcal{E}_{ki} \tilde{\partial}^k , \end{aligned} \quad (5.5)$$

with the background matrix \mathcal{E} defined as

$$\mathcal{E}_{ij} = G_{ij} + B_{ij} . \quad (5.6)$$

The DFT action (5.1) can be expressed as [9]

$$\begin{aligned} S_{\text{DFT}} &= \int d^D x d^D \tilde{x} \mathcal{L}_{\text{DFT}}(\mathcal{E}, \mathcal{D}, d) \\ &= \int d^D x d^D \tilde{x} e^{-2d} \left[-\frac{1}{4} G^{ik} G^{jl} G^{pq} (\mathcal{D}_p \mathcal{E}_{kl} \mathcal{D}_q \mathcal{E}_{ij} - \mathcal{D}_i \mathcal{E}_{lp} \mathcal{D}_j \mathcal{E}_{kq} - \overline{\mathcal{D}}_i \mathcal{E}_{pl} \overline{\mathcal{D}}_j \mathcal{E}_{qk}) \right. \\ &\quad \left. + G^{ik} G^{jl} (\mathcal{D}_i d \overline{\mathcal{D}}_j \mathcal{E}_{kl} + \overline{\mathcal{D}}_i d \mathcal{D}_j \mathcal{E}_{lk}) + 4 G^{ij} \mathcal{D}_i d \mathcal{D}_j d \right] . \end{aligned} \quad (5.7)$$

The rigid $O(D, D)$ symmetry $X' = h X$ acts as follows

$$\begin{aligned} \mathcal{E}' &= (\mathfrak{a} \mathcal{E} + \mathfrak{b})(\mathfrak{c} + \mathcal{E} \mathfrak{d})^{-1} , \\ \mathcal{D}_i &= M_i^j \mathcal{D}'_j , \quad \overline{\mathcal{D}}_i = \overline{M}_i^j \overline{\mathcal{D}}'_j , \end{aligned} \quad (5.8)$$

where the matrices M, \overline{M} are given by

$$M = (\mathfrak{d} - \mathfrak{c} \mathcal{E}^t)^t , \quad \overline{M} = (\mathfrak{d} + \mathfrak{c} \mathcal{E})^t . \quad (5.9)$$

As it will become relevant soon, we also provide the implied transformation of the metrics

$$G = \overline{M} G' \overline{M}^t . \quad (5.10)$$

The idea now is that the actions in the non-geometric frames correspond to different solutions of the strong constraint. The latter allows us to express the winding derivative in terms of the usual derivative. However, implementing this constraint and directly reducing the DFT action is not a trivial task so that we use the rigid $O(D, D)$ symmetry to rotate the solution of the strong constraint again to the simple form $\tilde{\partial} = 0$ and then perform the reduction.

Relation of DFT to non-geometric actions

To connect to our analysis from previous sections, our starting point is DFT with the fields $\widehat{\mathcal{E}} = \widehat{G} + \widehat{B}$ and an action $\mathcal{L}_{\text{DFT}}(\widehat{\mathcal{E}}, \widehat{\mathcal{D}}, \widehat{d})$. Now we are solving the strong constraint by an ansatz which contains the matrices used for the field redefinition of section 2.2 in the special case of constant a, b :

$$\widehat{\partial}^i = (b^t)^{ij} \partial_j, \quad \widehat{\partial}_i = (a^t)_i{}^j \partial_j. \quad (5.11)$$

Indeed the strong constraint (5.4) becomes

$$\partial_i A \partial_j B (b a^t + a b^t)^{ij} = 0. \quad (5.12)$$

Instead of reducing the DFT Lagrangian to a Lagrangian $\mathcal{L}(\widehat{\mathcal{E}}, \partial, \widehat{d})$ depending on only half of the coordinates we use the rigid $O(D, D)$ symmetry to transform it to a frame where the strong constraint is simply solved by $\widehat{\partial}^i = 0$. It will turn out below that the right choice for the $O(D, D)$ transformation is

$$h = (\mathcal{M}^t)^{-1} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \quad (5.13)$$

where \mathcal{M} is the $O(D, D)$ matrix we used for the field redefinitions and the definition of the anchor in the previous sections. The fields in the new frame are denoted as $\mathcal{E} = G + B$. Using (5.2) we find that the partial derivatives transform as follows

$$\begin{pmatrix} \widetilde{\partial} \\ \partial \end{pmatrix} = \mathcal{M} \begin{pmatrix} \widehat{\partial} \\ \widehat{\partial} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \widehat{\partial} \\ \widehat{\partial} \end{pmatrix}. \quad (5.14)$$

Therefore, the solution (5.11) to the strong constraint simply becomes $\widetilde{\partial}^i = 0$ in the new coordinates. Moreover, from (5.10) we get the relation between the old and the new metric

$$\widehat{G} = (a^t + (\widehat{G} - \widehat{B}) b^t) G (a^t + (\widehat{G} - \widehat{B}) b^t)^t, \quad (5.15)$$

which precisely agrees with the equations (2.23) and (2.24) for the field redefinition relating the geometric frame to the non-geometric one. The same can be shown for the two-form. This justifies the choice of the $O(D, D)$ -transformation (5.13). Since the dilaton d is invariant, the measure factor is

$$e^{-2d} = \sqrt{|G|} e^{-2\phi} = \sqrt{|\widehat{G}|} |\widehat{\gamma}^{-1}| e^{-2\phi}. \quad (5.16)$$

Therefore, we can conclude that reducing DFT in the new frame with $\widetilde{\partial}^i = 0$ results in the standard NS-NS action $S_{\text{NS-NS}}(G, B, \partial_i, \phi)$ with redefined background fields $G(\widehat{G}, \widehat{B})$ and $B(\widehat{G}, \widehat{B})$. But as was shown in section 4.3, this action is equivalent to the string action (4.40) in the non-geometric frame.

5.2 Relation to supergravity

After having considered the NS-NS sector to lowest order in α' , let us now turn to the remaining terms in the low-energy effective action of string theory. In the following we want to discuss how the constructions given above apply to Ramond-Ramond (R-R) and fermionic fields. It turns out that we can translate the whole supergravity action to the non-geometric frames.

General remarks

Let us recapitulate the necessary ingredients. In section 4 we have seen that any tensor in the standard frame becomes a ρ -tensor in the redefined theory if the anchor is applied to all the indices, cf. (4.27). Since the $O(D, D)$ induced field redefinition only concerns the metric and the B -field, any tensor invariant under B -gauge transformations becomes a ρ -tensor by anchoring. This applies in particular to fields which transform under gauge transformations different from B -gauge transformations, e.g. Ramond-Ramond fields. Hence we transform every (r, s) -tensor T on TM to an (r, s) - ρ -tensor \hat{T} on the Lie algebroid E via

$$\hat{T}^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s} = (\rho^{-1})^{\alpha_1}_{a_1} \dots (\rho^{-1})^{\alpha_r}_{a_r} (\rho^t)_{\beta_1}^{b_1} \dots (\rho^t)_{\beta_s}^{b_s} T^{a_1 \dots a_r}_{b_1 \dots b_s}. \quad (5.17)$$

Using (4.7) this also holds for covariant derivatives

$$\hat{\nabla}_\gamma \hat{T}^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s} = (\rho^t)_{\gamma}^c (\rho^{-1})^{\alpha_1}_{a_1} \dots (\rho^{-1})^{\alpha_r}_{a_r} \times (\rho^t)_{\beta_1}^{b_1} \dots (\rho^t)_{\beta_s}^{b_s} \nabla_c T^{a_1 \dots a_r}_{b_1 \dots b_s}, \quad (5.18)$$

which is just a special case of (5.17). Note that all terms appearing in the Lagrangian as well as in the equations of motion are tensors which do not transform under B -gauge transformations. Thus (5.17) and (5.18) suffice to translate every term. In addition, the measure transforms according to (4.37), so the appropriate determinant appears in the action. The following is the direct generalization of the results of [36].

R-R sector

The Ramond-Ramond fields in e.g. type IIA supergravity are the antisymmetric tensors C_1 and C_3 , with corresponding field strengths¹⁷

$$\begin{aligned} F_{a_1 a_2} &= 2 \nabla_{[a_1} C_{a_2]}, \\ F_{a_1 a_2 a_3 a_4} &= 4 (\nabla_{[a_1} C_{a_2 a_3 a_4]} + C_{[a_1} H_{a_2 a_3 a_4]}). \end{aligned} \quad (5.19)$$

¹⁷Note that they are usually defined with the partial instead of the covariant derivative, but the usual symmetric connection coefficients drop out in the antisymmetrization. Our ‘hatted’ Christoffel symbols however are in general not symmetric.

The Lagrangian for these fields is given by:

$$\begin{aligned} \mathcal{L}_{\text{IIA}}^{\text{R-R}} \sim & \frac{1}{4} F_{a_1 a_2} F^{a_1 a_2} + \frac{1}{48} F_{a_1 a_2 a_3 a_4} F^{a_1 a_2 a_3 a_4} \\ & - \frac{i}{144 \cdot \sqrt{|G|}} \epsilon^{a_1 \dots a_{10}} \nabla_{[a_1} C_{a_2 a_3 a_4]} \nabla_{[a_5} C_{a_6 a_7 a_8]} B_{a_9 a_{10}} , \end{aligned} \quad (5.20)$$

where $\epsilon^{a_1 \dots a_{10}}$ is the antisymmetric symbol (which is not a tensor). By applying (5.17), we can express this Lagrangian in the redefined fields on the Lie algebroid

$$\begin{aligned} \mathcal{L}_{\text{IIA}}^{\text{R-R}} = \hat{\mathcal{L}}_{\text{IIA}}^{\text{R-R}} \sim & \frac{1}{4} \hat{F}_{\alpha_1 \alpha_2} \hat{F}^{\alpha_1 \alpha_2} + \frac{1}{48} \hat{F}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \hat{F}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \\ & - \frac{i}{144 \cdot \sqrt{|g|}} \epsilon^{\alpha_1 \dots \alpha_{10}} \hat{\nabla}_{[\alpha_1} \hat{C}_{\alpha_2 \alpha_3 \alpha_4]} \hat{\nabla}_{[\alpha_5} \hat{C}_{\alpha_6 \alpha_7 \alpha_8]} \mathbf{b}_{\alpha_9 \alpha_{10}} , \end{aligned} \quad (5.21)$$

where the corresponding action includes the measure (4.37).

These redefined R-R fields are invariant under the usual gauge transformations

$$\begin{aligned} \delta_{\Lambda_{(0)}} \hat{C}_{\alpha_1} &= \hat{\nabla}_{\alpha_1} \Lambda , & \delta_{\Lambda_{(0)}} \hat{C}_{\alpha_1 \alpha_2 \alpha_3} &= -\Lambda \Theta_{\alpha_1 \alpha_2 \alpha_3} , \\ \delta_{\Lambda_{(2)}} \hat{C}_{\alpha_1 \alpha_2 \alpha_3} &= \hat{\nabla}_{[\alpha_1} \Lambda_{\alpha_2 \alpha_3]} , \end{aligned} \quad (5.22)$$

where $\Lambda_{(0)}$ and $\Lambda_{(2)}$ are arbitrary zero- and two-forms. The invariance under $\delta_{\Lambda_{(0)}}$ follows from the Bianchi identity (4.33).

Fermionic sector

In the following, Greek indices starting with μ will denote Lorentz indices, whereas Greek indices starting with α are Lie algebroid indices and Latin indices are TM indices, as before. To write down the Lie algebroid action for R-NS and NS-R fields we need to consider vielbein fields \hat{e}_α^μ which fulfill a relation analogous to the normal frame fields e_a^μ :

$$\hat{e}_\alpha^\mu \hat{e}_\beta^\nu g^{\alpha\beta} = \delta^{\mu\nu} = e_a^\mu e_b^\nu G^{ab}. \quad (5.23)$$

By (4.3), this implies $\hat{e}_\alpha^\mu = \rho^a_\alpha e_a^\mu$.

With these vielbeins, we can build a spin connection $\hat{\omega}$ on E using the standard formula:

$$\hat{\omega}_\gamma{}^\mu{}_\nu = \hat{e}_\alpha^\mu \hat{e}_\nu^\beta \hat{\Gamma}^\alpha_{\beta\gamma} + \hat{e}_\alpha^\mu D_\gamma \hat{e}_\nu^\alpha. \quad (5.24)$$

Using (4.11), we can write this as

$$\hat{\omega}_\gamma{}^\mu{}_\nu = \rho^c_\gamma \omega_c{}^\mu{}_\nu , \quad (5.25)$$

where

$$\omega_c{}^\mu{}_\nu = e_a^\mu e_\nu^b \Gamma^a_{bc} + e_a^\mu \partial_c e_\nu^a \quad (5.26)$$

is the standard spin connection on TM .

Now consider the fermionic sector of type IIA supergravity (which can be found in [43]). It contains the dilaton ϕ , the gravitino ψ_a , the dilatino λ , R-R fields F_{a_1, \dots, a_k} , covariant derivatives D_a and gamma matrices γ_a . The gamma matrices on TM are given by $\gamma_a = e_a^\mu \gamma_\mu$, so we can consistently define

$$\hat{\gamma}_\alpha = \hat{e}_\alpha^\mu \gamma_\mu = \rho^a{}_\alpha e_a^\mu \gamma_\mu = \rho^a{}_\alpha \gamma_a. \quad (5.27)$$

The kinetic term of the dilatino has the following form:

$$\mathcal{L}_{\text{IIA}}^\lambda \sim \bar{\lambda} \gamma^a \left(\partial_a - \frac{i}{4} \omega_a{}^{\mu\nu} \gamma_{\mu\nu} \right) \lambda, \quad (5.28)$$

where $\gamma_{\mu_1 \dots \mu_k} = \gamma_{[\mu_1} \dots \gamma_{\mu_k]}$. Because λ is a spinor, the spin connection ω has to be included. Since the dilatino does not have vector indices, we have $\hat{\lambda} = \lambda$, so we can write

$$\hat{\mathcal{L}}_{\text{IIA}}^{\hat{\lambda}} \sim \bar{\hat{\lambda}} \hat{\gamma}^\alpha \left(D_\alpha - \frac{i}{4} \hat{\omega}_\alpha{}^{\mu\nu} \gamma_{\mu\nu} \right) \hat{\lambda}. \quad (5.29)$$

The kinetic term of the gravitino looks like

$$\mathcal{L}_{\text{IIA}}^\psi \sim \bar{\psi}_a \gamma^{abc} \left(\nabla_b - \frac{i}{4} \omega_b{}^{\mu\nu} \gamma_{\mu\nu} \right) \psi_c \quad (5.30)$$

because ψ_a has a form index in addition to the (implicit) spinor indices.¹⁸ Here (5.17) reads $\hat{\psi}_\alpha = (\rho^t)_\alpha{}^a \psi_a$, and with the above the transformed Lagrangian is

$$\hat{\mathcal{L}}_{\text{IIA}}^{\hat{\psi}} \sim \bar{\hat{\psi}}_\alpha \gamma^{\alpha\beta\gamma} \left(\hat{\nabla}_\beta - \frac{i}{4} \hat{\omega}_\beta{}^{\mu\nu} \gamma_{\mu\nu} \right) \hat{\psi}_\gamma. \quad (5.31)$$

Again, the equivalence of the gravitino actions in both frames follows from (4.27).

5.3 Higher order corrections

The action (2.38) is the lowest order contribution in the string tension α' to the effective action of the massless modes G , B and ϕ . Although the higher order corrections are not unique due to a freedom of redefining the fields, all terms can be composed of (covariant derivatives of) the curvature tensor $R^a{}_{bcd}$, the three-form H , the dilaton $\partial_a \phi$ and contractions thereof [44, 45, 46].

For the translation of these higher order corrections to a non-geometric frame, we note that all terms in the action are scalars which are invariant under gauge

¹⁸Note that the Christoffel symbols drop out in the Lagrangian due to their symmetry. In the new frame, they are (in general) not symmetric, so we have to keep them (cf. footnote 17).

transformations of the Kalb-Ramond field. To obtain ρ -scalars on the Lie algebroid, we therefore just have to perform the replacements

$$\begin{aligned}
R^a{}_{bcd} &\rightarrow \widehat{R}^\alpha{}_{\beta\gamma\delta}, \\
H_{abc} &\rightarrow \Theta_{\alpha\beta\gamma}, \\
\partial_a\phi &\rightarrow D_\alpha\phi, \\
\sqrt{|G|} &\rightarrow \sqrt{|g|}|\rho^*|,
\end{aligned} \tag{5.32}$$

cf. (4.13), (4.34) and (4.37). Indeed, contractions of the latter fields are then ρ -scalars. The resulting terms contribute as higher order α' -corrections to the action (4.40) in the NS-NS sector, and are related by the general field redefinition (4.1) and (4.20) to the actions in the usual frame.

5.4 Equations of motion

The recipe applied above is also suitable for the equations of motion of the action (4.40). The explicit computation is very cumbersome, but we can equally well just transform the well-known equations of motion for (2.38). Again, every term therein is a gauge invariant tensor and anchoring it gives ρ -tensors. As the anchor is a bijection, we can just drop the overall anchor factors which yields an independent set of equations for \widehat{G} , \mathfrak{b} and ϕ . This way we obtain the equations of motion for the general redefined action (4.40)

$$\begin{aligned}
0 &= \widehat{R}_{\alpha\beta} + 2\widehat{\nabla}_\alpha\widehat{\nabla}_\beta\phi - \frac{1}{4}\Theta_{\alpha\mu\nu}\Theta_\beta{}^{\mu\nu}, \\
0 &= -\frac{1}{2}g^{\alpha\beta}\widehat{\nabla}_\alpha\widehat{\nabla}_\beta\phi + g^{\alpha\beta}\widehat{\nabla}_\alpha\phi\widehat{\nabla}_\beta\phi - \frac{1}{24}\Theta_{\alpha\beta\gamma}\Theta^{\alpha\beta\gamma}, \\
0 &= \frac{1}{2}\widehat{\nabla}^\mu\Theta_{\mu\alpha\beta} - (\widehat{\nabla}^\mu\phi)\Theta_{\mu\alpha\beta}.
\end{aligned} \tag{5.33}$$

Let us emphasize that (5.33) are the equations of motion for the action (4.40) in an arbitrary non-geometric frame. Here, we considered \mathfrak{b} instead of \widehat{B} for simplicity; the appearance of the former in (4.40) is analogous to B in (2.38).

5.5 Non-geometric frames – non-geometric backgrounds

The notion of non-geometry applies to string theory backgrounds which elude a description in terms of usual manifolds. In ordinary geometry, the transition functions between local patches of a manifold are diffeomorphisms, possibly accompanied by gauge transformations. These are encoded in the geometric group $G_{\text{geom}} = G_{d\Lambda} \rtimes G_{\text{diffeo}}$, the local symmetry group for the string action (2.38). For patching up non-geometric backgrounds, however, a transformation beyond G_{geom} is necessary. Hence, for identifying a non-geometric background global

properties have to be taken into account. Concrete examples of non-geometric backgrounds arise from T-dualizing geometric ones. The *T-fold* introduced in [16] gives such an example for which the structure group contains general $O(D, D; \mathbb{Z})$ transformations.

In the previous sections, we have given a description of string theory in general non-geometric *frames*. Here, different frames were defined by applying $O(D, D)$ transformations to a given generalized metric. The question now arises whether and how the description of a *given* non-geometric background might simplify by choosing an appropriate non-geometric frame. As one knows from the standard Q -flux background, the concrete expressions for the background fields might simplify, but the essential question is whether the transition functions can become members of the symmetry group in a non-geometric frame.

To analyze this question, let us consider the generalized metric (2.7). Suppose \mathcal{H}_1 and \mathcal{H}_2 are the generalized metrics in two overlapping patches of a non-geometric background with the transition function given by $\mathcal{T} \notin G_{\text{geom}}$

$$\mathcal{H}_1 = \mathcal{T}^t \mathcal{H}_2 \mathcal{T}. \quad (5.34)$$

Now, going to another frame by applying an $O(D, D)$ transformation \mathcal{M} to this background, the transition function \mathcal{T} changes to

$$\mathcal{T}' = \mathcal{M}^{-1} \mathcal{T} \mathcal{M}. \quad (5.35)$$

However, performing a field redefinition based on \mathcal{M} also changes the geometric group which, as we have seen, is the symmetry group of the action (4.40) in this non-geometric frame. The new symmetry group becomes $G'_{\text{geom}} = \mathcal{M}^{-1} G_{\text{geom}} \mathcal{M}$ so that

$$\mathcal{T}' \notin G'_{\text{geom}} \iff \mathcal{T} \notin G_{\text{geom}}, \quad (5.36)$$

i.e. the transition function remains to be non-geometric.

Q -flux example

As an example, we consider the approximate Q -flux background [12]. It arises from a three-torus parametrized by coordinates (x, y, z) with constant H -flux N by performing two T-dualities in the isometric directions, say x and y . The background is given by

$$G = \frac{1}{1 + N^2 z^2} (dx^2 + dy^2) + dz^2, \quad B = \frac{Nz}{1 + N^2 z^2} dx \wedge dy, \quad (5.37)$$

where we have set the radii of the torus to one. The z -direction is a cycle of the torus and as such admits a periodicity $z \mapsto z + k$ for $k \in 2\pi\mathbb{Z}$. However, the fields (5.37) are not periodic and the change in G and B cannot be compensated by a

diffeomorphism or a gauge transformation. Instead, the required transformation is given by a β -transform

$$\mathcal{T} = \begin{pmatrix} \mathbb{1} & K \\ 0 & \mathbb{1} \end{pmatrix} \quad \text{with} \quad K = \begin{pmatrix} 0 & -Nk & 0 \\ Nk & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.38)$$

and is not contained in G_{geom} . Performing the field redefinition (2.29) we obtain

$$\widehat{G} = dx^2 + dy^2 + dz^2, \quad \widehat{B} = -Nz \, dx \wedge dy. \quad (5.39)$$

In this frame the Q -flux background has a very simple form. In particular, the metric is well-defined and the B -field \widehat{B} just shifts by a constant as one moves around the z -cycle. This can be compensated by a simple gauge transformation $\widehat{B} \rightarrow \widehat{B} + Nk \, dx \, dy$. Besides the diffeomorphisms, the geometric group now contains the ρ -gauge transformations (4.28). Using the $O(D, D)$ -transformation (2.33) as well as (5.35), the ρ -gauge transformations in G'_{geom} and the transition matrix (5.38) in the new frame read

$$\mathcal{M}'_{\text{B}} = \begin{pmatrix} \mathbb{1} & \widehat{G}^{-1} \mathbf{B} \widehat{G}^{-1} \\ 0 & \mathbb{1} \end{pmatrix} \quad \text{and} \quad \mathcal{T}' = \begin{pmatrix} \mathbb{1} & 0 \\ \widehat{G} K \widehat{G} & \mathbb{1} \end{pmatrix}, \quad (5.40)$$

respectively. Clearly, $\mathcal{T}' = \mathcal{M}'_I \mathcal{T} \mathcal{M}_I$ is *not* an element of the transformed geometric group G'_{geom} .¹⁹ Equivalently, we observe that constant shifts in \widehat{B} are not exact with respect to the redefined exterior derivative (4.15). This shows that although a field redefinition is able to cast a non-geometric background into a simple form with transformations reminiscent of the usual symmetries, it cannot provide a global description.²⁰

To summarize, the framework we have developed can describe non-geometric backgrounds patch-wise. If patching up requires a transformation beyond G_{geom} , different patches are still described by different actions (4.40).

6 Conclusions

In this paper, we have elaborated on the non-geometric part of generalized geometry, that is, the consequences of the existence of β -transformations. We found a remarkable rich structure, which we connected to the mathematical theory of Lie algebroids. This provides a general framework to study non-geometric backgrounds, in which former studies of non-geometric actions appear as two specific examples.

¹⁹Note that in this example diffeomorphisms in G'_{geom} are the same as in G_{geom} .

²⁰In [32], in addition to the field redefinition (2.29) the further constraint $\beta^{ij} \partial_j () = 0$ was implemented which truncates the action such that (5.38) becomes a proper symmetry.

We observed that a β -transform, i.e. an $O(D, D)$ transformation which is not in the geometric group, naturally gives rise to a field redefinition of the metric and the Kalb-Ramond two-form. Expressing the string action in these new variables, we identified the organizing principle for the many resulting terms as the differential geometry of certain Lie algebroids. The latter could be defined via an anchor, mapping either the tangent or the co-tangent space to the standard tangent bundle. The data of the anchor could be read off directly from the $O(D, D)$ -transformation. Particularly for β -transforms, the Lie algebroid was not simply related to a choice of non-holonomic basis, but gives an unprecedented branch of differential geometry. Note that in this latter sense, non-geometric frames are still geometric.

At the heart of the paper, in a general setting we proved the connection between the field redefined action and the action expressed in terms of objects appearing in the differential geometry of the associated Lie algebroid. Moreover, we established how diffeomorphisms as well as gauge transformations carry over from usual Riemannian geometry to the non-geometric side. The behavior under diffeomorphism originated from the very general construction of the differential geometry of the underlying Lie algebroid, where function-linearity was built in. Gauge transformations were more subtle as redefining with the anchor introduced a gauge dependence in every object. To distinguish this overall gauge dependence from an inherent gauge dependence, we introduced the notion of a ρ -tensor.

We also related our non-geometric actions to double field theory. More concretely, we showed how, for rigid $O(D, D)$ transformations, the different non-geometric frames are related to different solutions to the strong constraint in DFT. We confirmed that after implementing this solution, DFT gives indeed our non-geometric actions. It was fairly straightforward to generalize the construction also to the additional terms appearing in the effective action of superstring theory, i.e. the R-R and fermionic terms. In addition, we pointed out that higher α' -corrections can also be described in the non-geometric frames.

What might appear a bit disillusioning is that these non-geometric frames do only provide a good description of global non-geometric backgrounds in each patch. We have seen that performing a non-geometric field redefinition might bring the metric and the two-form into a very simple form. However, the transition functions of non-geometric T-fold backgrounds, by definition, involve β -transforms (i.e. T-duality transformations), which are not in the symmetry group of the action in a specific non-geometric frame. In other words the string action in two patches glued together by a β -transform cannot be described by a single non-geometric action. Contrarily, in DFT the additional winding dependence in the diffeomorphisms and winding diffeomorphisms allows such a global description.

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A Invertibility of the anchor γ

In this appendix we comment on the invertibility of γ by considering the generators of $O(D, D)$. The Lie subgroup generated by the matrices in table 1 has dimension $2 \cdot \frac{D(D-1)}{2} + D^2 = 2D^2 - D$, which is the dimension of $O(D, D)$. Thus, this already is the identity component $O(D, D)_0$. (Remember that a connected Lie group is generated by any open neighborhood of the identity.) The quotient group $O(D, D)/O(D, D)_0 = \pi_0(O(D, D)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by the following transformations:

$$\mathcal{M}_\pm = \begin{pmatrix} \mathbb{1} - E_1 & \pm E_1 \\ \pm E_1 & \mathbb{1} - E_1 \end{pmatrix}, \quad (\text{A.1})$$

where $E_1 = \text{diag}(1, 0, \dots, 0)$. Thus, a general $O(D, D)$ matrix \mathcal{M} is a (finite) product of such transformations:

$$\mathcal{M} = (\mathcal{M}_+)^{\eta_+} (\mathcal{M}_-)^{\eta_-} \prod_{i=1}^n \mathcal{M}_{\beta_i} \mathcal{M}_{B_i} \mathcal{M}_{A_i}, \quad (\text{A.2})$$

where $\eta_\pm \in \{0, 1\}$. To find the frame corresponding to \mathcal{M} , we can apply the corresponding field redefinitions successively.

Now we just have to show that the anchor corresponding to each generator is invertible, so we consider

$$\begin{aligned} \gamma_{\text{diffeo}} &= (\mathbf{A}^t)^{-1}, \\ \gamma_B &= \mathbb{1}, \\ \gamma_\beta &= \mathbb{1} - (G - B)\beta, \\ \gamma_\pm &= \mathbb{1} - E_1 \pm (G - B)E_1. \end{aligned} \quad (\text{A.3})$$

For the first two, invertibility is obvious. Note that $O(D, D)$ transformations only act on the (Euclidean) compact part of the spacetime manifold. Thus, we should be aware that the non-trivial components of the anchor above only refer to the internal manifold, where G is positive definite. Then $(G - B)$ is positive definite (in the sense that its Hermitean part is), and all such matrices have a positive definite inverse. So, $\gamma_\beta = (G - B)[(G - B)^{-1} - \beta]$ is invertible as well. (Recall that B and β are antisymmetric.)

In order to show that γ_\pm is invertible, we note that

$$\det(\gamma_\pm) = \pm G_{11} = \pm \langle e_1, G e_1 \rangle \neq 0, \quad (\text{A.4})$$

where $e_1 = (1, 0, \dots, 0)^t$.

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